DIFFERENTIAL EQUATIONS

Structure

11.1 INTRODUCTION

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions–the set of functions that satisfy the equations. Only the simplest differential equations are solvable by explicit formulas, however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

11.2 OBJECTIVES

After reading this unit you should be able to -

- Differential Equations and their concepts.
- Exact Equations and their concepts.

11.3 DIFFERENTIAL EQUATION

Differential Equation. An equation which involves differentials or differential coefficient (i.e. derivatives) is called a differential equation.

Ordinary differential equations are those which involve only one independent variable and the derivatives or different w.r.to it, e.g. $\frac{dy}{dx} = 5x + 7$ is an ordinary differential equation.

Partial differential equations are those which involve two or more independent variables and the partial derivatives w.r. to any of them.

e.g. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ is partial differential equation.

Order of differential equation. The order of a differential equation is the order of the highest derivative in the equation.

e.g., $\frac{dy}{dx} = 2x, \frac{d^2y}{dx^2} = -ny$ are of order one and two respect.

Degree of a differential equation. The degree of a differential equation is the degree of the highest derivative when the equation has been made free from radical and fraction as far as the derivatives are concerned.

e.g., $\frac{dy}{dx} = 2x$ is of first degree and first order.

Note. Differential equation is of continuous variable :

A first order linear differential equation generally take the form

$$\frac{dy}{dt} + u(t)y = w(t)$$

where u and w are two functions of t, as in y. In contrast to dy/dt and y, however, no restriction whatsoever is placed on the independent variable t.

The Homogeneous Case

If u and w are constant functions and if w happens to be identically zero, will become

$$\frac{dy}{dt} + ay = 0, \text{ where a is some constant.}$$

This differential equation is said to be homogeneous on account of the zero constant term.

Thus, above equation can be written atternatively as

$$\frac{1}{y}\frac{dy}{dt} = -a$$

- The solution of this equation will be,

  $y(t) = Ae^{-at}$ (general solution)

  $y(t) = y(o)e^{-at}$ (definite solution)

Two things about the solutions of a differential equation.

(1) The solution is not a numerical value, but rather a function $y(t)$ – ..a time path if t symbolizes times,

(2) the solution $y(t)$ is free of any derivative or differential expressions, so that as soon as a specific value of t is substituted into it, a corresponding value of y can be calculated easily.

Q. $\frac{dy}{dt} - 2y = 0$ **, with the initial condition** $y(0) = 9$

Solution. $\frac{dy}{dt} - 2y = 0$

Since, $a = -2$, the complementary function can be expressed simply as,

$yc = Ae^{-(-2)t}$

$yc = Ae^{2t}$

 A particular function

$yp = 0/-2 = 0$

Total solution,

$y(t) = yc + yp$

$y(t) = Ae^{2t} + 0$

$y(t) = Ae^{2t}$ ...(1)

 $y(0) = 9$, Put in (1), we get

$y(0) = Ae^{2(0)}$

$9 = Ae^{0}$

$\boxed{A = 9}$

 General solution

$y(t) = 9\, e^{2t}$

Q. $\frac{dy}{dt} - 5y = 0;\ y(0) = 6.$

Solution. $y(t) = 6e^{5t}$

Note. *In case a < 0 then, system is unstable (diverge)*

In case a > 0, then system is stable then it converges.

The Non-Homogeneous case

When non-zero constant takes the place of the zero we have a non-homogeneous linear differential equation $\frac{dy}{dt} + ay = b$.

The solution of this equation will consist of the sum of two terms, one of which is called the complementary function (*i.e., yc*) and the other known as the particular integral (*i.e, yp*).

□ $yc = Ae^{-at}$.

Since , the particular integral is any particular solution of the complete equation. The particular solution will be,

$yp = b/a \ (a \neq 0)$.

The, sum of *yc* and *yp* constitute the general solution of the complete equation.

$$y(t) = yc + yp = \boxed{Ae^{-at} + b/a}$$

For definite solution, we take $y(0)$ when $t = 0$, Then by setting $t = 0$,

$y\,(0) = A + b/a$, and $A = y(0) - b/a$

□ Definite solution, will be

$$y(t) = \left[y_0 - \frac{b}{a}\right]e^{-at} + b/a$$

Q. Solve the equation $\frac{dy}{dt} + 2y = 6$, with the initial condition y(0) = 10.

Solution. The given equation is,

$\frac{dy}{dt} + 2y = 6,$ Here, $a = 2,\ b = 6$

For complementary solution, we have

$yc = Ae^{-at}$

□ $yc = Ae^{-2t}$

For particular solution, we have

$yp = + b/a$

$= + 6/2$

$\therefore$ $y_p = 3$

$\therefore$ Complete solution will be,

$y(t) = y_c + y_p$

$y(t) = Ae^{-2t} + 3$

Since, $y(0) = 10$,

$\therefore$ Putting $t = 0$ and $y(t) = 10$, we get

$$10 = Ae^{-2(0)} + 3$$

$$10 = Ae^{0} + 3$$

$$10 = A + 3$$

$$\therefore A = 10 - 3$$

$$\boxed{A = 7}$$

$\therefore$ Definite solution

$y(t) = 7e^{-2t} + 3$

Solve it.

Q. $\frac{dy}{dt} + 4y = 12$, $y(0) = 2$, **Ans.** $y(t) = -e^{-4t} + 3$

Q. $\frac{dy}{dt} + 10y = 15$, $y(0) = 0$ **Ans.** $y\ (+) = \frac{3}{2}(1 - e^{-10})$

Q. $2\frac{dy}{dt} + 4y = 6$; $y(0) = 1\frac{1}{2}$ **Ans.** $y(t) = 3/2$

Q. $\frac{dy}{dt} + y = 4$; $y(0) = 0$, **Ans.** $y(t) = 4\ (1 - e^{-t})$

Q. $\frac{dy}{dt} - 7y = 7$, $y(0) = 7$; **Ans.** $y(t) = 8e^{7t} - 1$

In case if there is a = 0

The differential equation in that case will be,

$$\frac{dy}{dt} = b,$$

By straight integration, its general solution can

$y(t) = bt + c..,$ c is an arbitrary constant.

Since, $a = 0$, the complementary function will be,

$yc = Ae^{-at} = Ae^{\circ} = A$ (A = an arbitrary constant)

As to the particular solution, will be

$$yp = bt$$

 The general solution is therefore

$$y(t) = yc + yp$$

 $y(t) = A + bt$

yc is a constant whereas yp is a function of time,

We can definite solution to be,

i.e. for, if y (0)

$$y(t) = y(0) + bt$$

Q. $\frac{dy}{dt} = 23; y(0) = 1.$

Solution. $\frac{dy}{dt} = 23.$

 the complementary solution is,

$$yc = A$$

 the particular solution is,

$$yp = 23\ t$$

 the general solution will be

$$y(t) = A + 23t \qquad ...(1)$$

□ Since $y(0) = 1$, □ equation (1) becomes

$1 = A + 23\ (0)$

$1 = A + 0$

□ $\boxed{A = 1}$

□ the definite solution is,

$y(t) = 1 + 23\ t$

Q. $\frac{dy}{dt} = 2, y(0) = 5$ **Ans.** $y(t) = 5 + 2t$

Q. $\frac{dy}{dt} = 50, y(0) = 2$ **Ans.** $y(t) = 2 + 50\ t$

11.4 VARIABLE COEFICIENT AND VARIABLE TERM

In the more general case of a first order linear differential equation.

$$\frac{dy}{dt} + u(t)y = w(t).$$

where, $u(t)$ and $w(t)$ represents a variable coefficient and a variable term, respectively.

Homogeneous Case.

Here, the differential equation is of type,

$$\frac{dy}{dt} + u(t)y = 0, \quad \text{or}, \quad \frac{1}{y}\frac{dy}{dt} = -u(t)$$

□ By integrating both sides in turn with respect t.

$$\text{Left side} = \int \frac{1}{y}.\frac{dy}{dt}.dt = \int \frac{1}{y}.dy = \log y + c$$

$$\text{Right side} = \int -u(t).dt = -\int u(t).dt$$

∴ When the two sides are equated, the result is,

$$\log y = -c - \int u(t).dt.$$

Then, the desired path can be obtained by taking

$$y(t) = e^{\log y} = e^{-c} \cdot e^{-\int u(t).dt}$$

$$= Ae^{-\int u(t).dt} \qquad \text{where } A = e^{-c}$$

This is general solution of the differential equation.

Q. $\frac{dy}{dt} + 3t^2y = 0$, find the general solution.

Solution. $\frac{dy}{dt} + 3t^2 y = 0$

Here, $u = 3t^2$

and $\int u.dt = \int 3t^2.dt = 3\frac{t^3}{3} + c = t^3 + c.$

∴ The general solution is

$$y(t) = Ae^{-(t^3+c)} \qquad \{\text{where, } A = e^{-c}\}$$

$$= Ae^{-t^3}.e^{-c}$$

$$= B.e^{-t^3} \qquad \text{where } B = Ae^{-c}$$

Q. $\frac{dy}{dt} + 2ty = 0$ **Ans.** $Ae^{-(t^2+c)} = Be^{-t^2}$

Q. $\frac{dy}{dt} + 7t = 0$ **Ans.** $Ae^{-(7t^{2/2}+c)} = Be^{-\frac{7t^2}{2}}$

Non-Homogeneous Case :

(*i.e.*) the differential equation is of the form

$$\frac{dy}{dt} + u(t)y = w(t).$$

$\therefore$ The general solution is

$$y(t) = e^{-\int u.dt.}\left(A + \int we^{\int udt}.dt\right)$$

Q. Find the general solution of the equation

$$\boldsymbol{\frac{dy}{dt} + 2ty = t},$$

Solution. Here, $\frac{dy}{dt} + 2ty = t,$

Here $u = 2t,\ w = t,$ and $\int u.dt = t^2 + c.$

$\therefore$ General solution $y(t) = e^{-(t^2+c)}\left(A + \int t.e^{t^2+c}.dt\right)$

$$= e^{-t2}.e^{-c}(A+e^{c})t.e^{t2}.dt)$$

$$= A.e^{-t^2}.e^{-c} + e^{-t^2}\left(\frac{1}{2}e^{t^2} + c\right)\left[e^{-c}e^{c} = 1\right]$$

$$\therefore \left(Ae^{-k} + c\right)e^{-t^2} + \frac{1}{2}$$

$= Be^{-t^2} + \frac{1}{2}$. where $B = Ae^{-k} + c$

Solve it.

Q. Find the general solution $\boldsymbol{\frac{dy}{dt} + 4ty = 4t}$

Solution. $Ae^{-2t^2}+1$

Q. $\dfrac{dy}{dt}+5y=15$ **Ans.** $y(t) = Ae^{-5t} + 3$

Q. $\dfrac{dy}{dt}+2ty=t; y(0)=\dfrac{3}{2}$ **Ans.** $y(t)=e^{2t}+\dfrac{1}{2}$

Q. $2\dfrac{dy}{dt}+12y+2e^{t}=0$, $y(0)=\dfrac{6}{7}$ **Ans.** $y(t) = e^{-6t}-\dfrac{1}{7}e^{t}$

Q. $x\dfrac{dy}{dx}-ay=x+1$, **Ans.** $\dfrac{x}{1-a}-\dfrac{1}{a}+cx^{9}$

11.5 EXACT DIFFERENTIAL EQUATION

The equation Mdx + Ndy = 0 is said to be exact if Mdx + Ndy is the exact differential of function of x and y, say u, *i.e.,*

$$M.dx + Ndy = 0$$

The necessary and sufficient condition that the equation.

Mdx + Ndy = 0, may be exact is that $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ and its solution is given by

$\int M dx+\int$(terms of N free from x).$dy = c$

y constant

Q. Solve $(x^2 - y^2 - 1)$. $dx - 2xy.dy = 0$.

Solution. Given $(x^2 - y^2 - 1)dx - 2xy\ dy = 0$

Compare with Mdx + Ndy = 0,

$M = x^2 - y^2 - 1$, $N = -2xy$

$$\frac{\partial M}{\partial y} = 0 - 2y, \quad \frac{\partial N}{\partial x} = -2y \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given differential equation is an exact differential equation. Its solution is

$$\int_{y\ \text{constant}} M\,dx + \int (\text{terms of N free from } x)\,dy = c$$

$$\int_{y-\ \text{constant}} (x^2 - y^2 - 1).dx + \int 0.dy = c$$

$$\frac{x^3}{3} - y^2x - x + k = c$$

$$\Rightarrow \frac{x^3}{3} - y^2x - x = c - k$$

$$\Rightarrow x^3 - 3y^2x - 3x = c_1 \quad \{\text{where, } c_1 = 3(a - k)$$

Solve it

Q. $(2y^2 + 4xy - x^2)\,dx + (2x^2 + 4xy - y^2) = dy = 0$

Ans. $-x^3 - y^3 + 6y^2x + 6yx^2 = c$

Q. $(x^2 - 2xy + 3y^2)\,dx + (4y^3 + 6xy - x^2)\,dy = 0$

Ans. $x^3 - 3x^2y + 9y^2x + 3y^4 = c_1$

Q. $(x^2 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right).dy = 0$

Ans. $x^3 + 3ye^{x/y} = c_1$

Q. $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)dy = 0$

Ans. $x + ye^{x/y} = c$

Q. $(1 + xy)\,ydx + (1 - xy)x.dy = 0$

Ans. $\log(x/y) - \log k = \frac{1}{xy}$

11.6 HIGHER ORDER DIFFERENTIAL EQUATIONS

The relevant differential equation

$$y''(t) + a_1 y'(t) + a_2 y = b$$

or $$\frac{d^2 y}{dt^2}(a) + a_1 \frac{dy}{dt} + a_2 y = b \qquad ...(1)$$

where, a_1, a_2 and b are all constants. If the term b is identically zero, we have a homogeneous equation, but if b is a non zero constant, the equation, is non-homegeneous.

The particular solution (Integral) (yp) :

yp can be found by :

$$y'(t) = y''(t) = 0$$

so that equation becomes.

$a_2 y = b$ with the solution $y = b/a_2$.

$\therefore$ The desired particular solution will be

$$yp = b/a^2$$

The complementary function will be (yc)

The general equation will be reduced to

$$y''(t) + a_1 y'(t) + a_2 y = 0$$

(*i.e.*) $$y''(t) + a_1 y'(t) + a_2 y = 0 \qquad ...(1)$$

Here put $y'(t) = r\text{A}e^{rt}$, $y''(t) = r^2\ \text{A}e^{rt}$, $y = \text{A}e^{rt}$

(*i.e.*) equation (2) becomes

$$r^2 . \text{A}e^{rt} + ar\text{A}e^{rt} + a_2\ \text{A}e^{rt} = 0$$

$$\therefore \text{A}e^{rt} + (r^2 + ra_1 + a_2) = 0$$

$$\therefore r^2 + a_1 r + a_2 = 0$$

$\therefore$ the characteristic roots will be,

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$\therefore$ The values of these two roots, we assign to r,

in the solution, $y = Ae^{rt}$

(i.e.) $y_1 = A_1 e^{r_1 t}$ and $y_2 = A_2 e^{r_2 t}$

where, A_1, A_2 are two arbitrary constants and r_1 and r_2 are the characteristic roots.

Case I. (Distinct real roots) :

Let r_1 and r_2 will take distinct real value.

Then,

$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$ (*i.e.* $r_1 \neq r_2$)

Note. The equilibrium is stable if and only if $r_1 < 0$ and $r_2 < 0$

Case II. Repeated real roots ($r_1 = r_2$). Then,

$$yc = A_1 e^{rt} + A_2 e^{rt} = (A_1 + A_2 t)e^{rt}$$

Note. Equilibrium is stable if $r < 0$, and it converges.

Case III (Complex Roots)

$$yc = A_1 e^{(h+vi)t} + A_2 e^{(h-vi)t}$$

$$= e^{ht}(A_1 e^{vit} + A_2 e^{-vit})$$

where $h = -\frac{a_1}{2}$, and $v = \frac{\sqrt{4a_2 - a_1^2}}{2}$

$\therefore$ $yc = e^{ht}[(A_1 + A_2)\cos vt + (A_1 - A_2)\sin vt]$

or $yc = e^{ht}[A_5 \cos vt + A_6 \sin vt]$

$A_5 = A_1 + A_2$, $A_6 = A_1 - A_2$.

if $\quad h < 0 \quad \Rightarrow$ condition of dynamic stability.

if $\quad h > 0 \quad \Rightarrow$ condition of dynamic unstability.

In case Q is of the form e^{ax}.

then $\quad yp = \dfrac{1}{f(D)} e^{ax},$

$$= \frac{1}{f(a)} . e^{ax} \quad \text{if } f(a) \neq 0$$

when b is e^{ax} *and* $\quad f(a) = 0$, then

$$yp = x. \frac{1}{f'(0)} . e^{ax}$$

$$yp = x \frac{1}{f'(a)} e^{ax}$$

Q. Solve $\boldsymbol{\dfrac{d^2y}{dx^2} - 6 \dfrac{dy}{dx} + 9y = 0}$

Sol. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0 \quad ...(1)$

The particular solution can be found by,

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} = 0$$

equation (1) will become

$$0 - 6\ (0) + 9y = 0$$

$$9y = 0$$

$$y = 0$$

i.e. $\quad \boxed{yp = 0}$

$\therefore$ Complementary solution, $y = Ae^{rx}, \frac{dy}{dx^2} = rAe^{rx}$

Put, $\frac{d^2y}{dx^2} = r^2Ae^{rx}$

$\therefore$ Put it in (1)

$$r^2 Ae^{rx} - 6rAe^{rx} + 9Ae^{rx} = 0$$

or, $r^2 - 6r + 9 = 0$

$\therefore r = 3, 3$

$\therefore yc = e^{3x} (A_1 + A_2x) \quad y(0)=1$

$\therefore y(x) = e^{3x} (A_1 + A_2x)$

Q. Solve $\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 40y = 0$

Sol. Here,

$yp = 0$

$\therefore$ For complementary solution

$$y = Ae^{rx}, \frac{dy}{dx} = rAe^{rx}, \frac{dy^2}{dx} = r^2Ae^{rx}$$

$\therefore$ Put it in (1), we get,

$$r^2 + 3r - 40 = 0$$

$$r_1, r_2 = -8, 5$$

$\therefore \quad yc = c_1e^{-8x} + c_2e^{5x}$

The general solution, $y(x) = c_1e^{-8x} + c_2e^{5x}$,

Q. $\boldsymbol{\frac{d^2y}{dt^2} + \frac{4dy}{dt} + 13y = 0}$

Solution. Here,

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$$

 Its particular solution is

$$yp = 0$$

For complementary solution is

Put $y = Ae^{rt}, \frac{dy}{dt} = rAe^{rt}, \frac{dy^2}{dt^2} = r^2Ae^{rt}$

 $r^2Ae^{rt} + 4r\ Ae^{rt} + 13\ Ae^{rt} = 0$

 $r^2 + 4r + 13 = 0$

 $r = \frac{-4 \pm \sqrt{16 - 41:13}}{2.1}$

 $r = -2 \pm 3i$

General solution

 $y = e^{-2x}(c_1 \cos 3t + c_2 \sin 3t)$

Solve it :

Q. $\frac{2d^2y}{dt^2} + \frac{5dy}{dt} - 12y = 0$ **Ans.** $y = c_1{}^{e-4t} + c_2e^{3/2}t$

Q. $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + 17y = 34$ **Ans.** $y(t) = e^{-t} (\cos 4t + \sin 4t) + 2.$

Q. $\frac{d^2y}{dt^2} + \frac{6dy}{dt} + 9y = 27$ **Ans.** $y(1)\ A_1e^{-3t} + A_2te^{-3t} + 3$

If $y(0) = 5, y'(0) = -5, \ y(t) = 2e^{-3t} + te^{-3t} + 3$

Q. $\boldsymbol{y''(t) + y'(t) - 2y = 0}$

Sol. $y(t) = A_{1e^t} + A_{2e^{-2t}}$

Q. Find the general solution of $y''(t) + 3y'(t) - 4y = 12$ and then general solution $y(0) = 4$, $y'(0) = 2$

Solution. The given equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = 12 \qquad ...(1)$$

∴ For particular solution,

$$yp = \frac{12}{-4} = -3$$

∴ For complementary solution,

$$\text{Put, } y = Ae^{rt}, \ \frac{dy}{dt} = rAe^{rt}, \ \frac{d^2y}{dt^2} = r^2Ae^{rt}$$

∴ equation (1), we get

$$r^2 + 3r - 4 = 0$$

$$∴ (r - 1)(r \pm 4) = 0$$

$$∴ r = 1, -4$$

$$∴ = A_1e^t + A_2e^{-4t}$$

For general solution,

$$y(t) = A_1e^t + A_2e^{-4t} - 3$$

$$y(0) = A_1e^0 + A_2e^0 - 3$$

$$4 = A_1 + A_2 - 3$$

$$4 + 3 = A_1 + A_2$$

$$\boxed{7 = A_1 + A_2} \qquad ...(2)$$

$$y'(t) = A_1e^t - 4A_2e^{-4t}$$

$y'(0) = 2,$

$2 = A_1e^0 - 4A_2e^0$

$\boxed{2 = A_1 - 4A_2}$ ...(3)

Finding, we get $A_1 = 6$, and $A_2 = 1$

∴ Definite solution will be

$$y(t) = 6e^t + e^{-4t} - 3$$

Q. $y''(t) + 6y'(t) + 9y = 27,\ y(0) = 5,\ y'(0) = 5$

Ans. $y(t) = 2e^{-3t} + te^{-3t} + 3.$

Q. $y''(t) - 2y'(t) + 17y = 34,\ y'(0) = 3,\ y'(0) = 11,$

Ans. $y(t) = e^{-t}(\cos 4t + 3\sin 4t) + 2$

Q. $y''(t) - 2y'(t) + y = 3, y(0) = 4,\ y'(0)$

Ans. $y(t) = e^t + te^t + 3$

Q. $y''(t) - 4y'(t) + 8y = 0,\ y(0) = 3,\ y'(0) = 7,$

Ans. $y(t) = e^{2t}(3 \cos 2t + 1/2 \sin 2t).$

Q. $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = e^{4x}$

Sol^n The given equation is

$$\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = e^{4x}$$

∴ For complementary equation. $\frac{dy}{dx} = rAe^{rt}, y = Ae^{rt}$

Put $\frac{d^2y}{dx^2} = r^2Ae^{rt}$

i.e. $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 0$

i.e. $r^2Ae^{rt} - 5rAe^{rt} + 6Ae^{rt} = 0$

i.e. $(r^2 - 5r + 6)\, Ae^{rt} = 0$

i.e. $r^2 - 5r + 6 = 0$

$\therefore \quad r = 2,3.$

$\therefore \quad yc = A_1e^{2x} + A_2e^{3x}$

Now, particular solution : $(D^2 - 5D + 6)\, y = e^{4x}$.

$$yp = \frac{e^{4x}}{f(D)}$$

$$= \frac{e^{4x}}{D^2 - 5D+6} = \frac{e^{4x}}{4^2 - 5.4 + 6} = \frac{e^{4x}}{2}$$

Complete solution is,

$$y(t) = yc + yp$$

$$y(t) = A_1e^{2x} + A_2e^{3x} + \frac{e^{2x}}{2}$$

Q. $\boldsymbol{\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{2x}}$

Sol. $(D^2 - 6D + 8)\, y = e^{2x} \left[D = \frac{d}{dx}\right]$

For complementary solution

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

Now $y = Ae^{rt}$, $\dfrac{dy}{dx} = rAe^{rt}$, $\dfrac{d^2y}{dx^2} = r^2Ae^{rt}$

□ $r^2 Ae^{rt} - 6rAe^{rt} + 8Ae^{rt} = 0$

□ $(r^2 - 6r + 8)Ae^{rt} = 0$

□ $r^2 - 6r + 8 = 0$

□ $r = 2, 4$

$yc = A_1e^{2x} + A_2e^{4x}$

Particular Solution

$$yp = \frac{e^{2x}}{D^2 - 6D + 8}$$

$$= \frac{e^{2x}}{2^2 - 6(2) + 8}$$

$$= \frac{e^{2x}}{4 - 12 + 8}$$

Since, the denominator is zero, the differentiation

$$= \frac{x.e^{2x}}{2D - 6}$$

$$= \frac{x.e^{2x}}{2(2) - 6}$$

$$= \frac{x.e^{2x}}{4 - 6}$$

$$= \frac{xe^{2x}}{-2}$$

$$= \frac{-xe^{2x}}{2}$$

 Complete solution will be,

$$y(t) = A_1e^{2x} + A_2e^{4x} - \frac{x}{2}e^{2x}$$

Q. $\boldsymbol{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + dy = 3e^{5x+2}}$

Sol. The given solution.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \frac{dy}{dx} = 3e^{5x+2}$$

The complementary solution

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx}y = 0$$

 $r^2Ae^{rt} - 2rAe^{rt} + Ae^{rt} = 0$

 $r^2 - 2r + 1 = 0$

 $r = 1, 1$

 $yc = (A_1 + A_2x)e^x$

 Particular integral, will be,

$$yp = 3\frac{1}{D^2 - 2D+1}.e^{5x+2}$$

$$= \frac{3}{D^2 - 2D+1}.e^{5x}.e^2$$

$$= e^2.\frac{3}{D^2 - 2D+1}.e^{5x}$$

$$= e^2 . \frac{3}{5^2 - 2(5) + 1} . e^{5x}$$

$$= e^2 . \frac{3}{25^{-12+1}} . e^{5x}$$

$$= \frac{3e^2 . e^{5x}}{16}$$

$$= \frac{3e^{5x+2}}{16}$$

 $y(t) = (A_1 + A_2 x)e^x + \frac{3}{16} . e^{5x+2}$

Q. $\boldsymbol{\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 3e^{2x} + 4}$

Sol. Given

$$\frac{d^2 y}{dx^2} - \frac{5dy}{dx} + 6y = 3e^{2x} + 4$$

 For complementary solution

$$\frac{d^2 y}{dx^2} - \frac{5dy}{dx} + 6y = 0$$

 $yc = A_1 e^{2x} + A_2 e^{3x}$

 Particular solution,

$$= \frac{1}{D^2 - 5D + 6} . (3e^{2x} + 4)$$

$$= 3\frac{1}{D^2 - 5D + 6} . e^{2x} + 4 . \frac{1}{D^2 - 5D + 6} . e^{0x}$$

$$= 3.\frac{x}{DD - 5} + 4.\frac{e^{0x}}{0- 0+6}$$

$$= \frac{3x.e^{2x}}{-1} + \frac{4}{6}$$

$$= - 3xe^{2x} + 2/3$$

$$\therefore y(t) = A_1e^{2x} + A_2e^{3x} - 3xe^{2x} + 2/3$$

Solve it :

Q. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = e^x$

Ans. $(A_1 + A_2x)e^{4x} + \frac{1}{9}e^x$

Q. $\frac{d^2y}{dx^2} - a^2.y = e^{ax} + e^{nx}$

Ans. $C_1e^{ax} + C_2e^{-ax} + \frac{xe^{ax}}{2a} + \frac{e^{nx}}{n^2 - a^2}$

Q. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = e^{3x}$

Sol. $y = A_1e^{5x} + A_2e^{-x} - \frac{e^{3x}}{8}$

11.7 SUMMARY

We end this lesson by summarizing what we have covered in it.

i) Differential Equations and their concepts.

ii) Exact Differential Equations.

iii) Differential Equations of lower order and higher order.

11.8 LESSON END EXERCISE

Q. 1. Explain the procedure for the solution of the linear differential equation ?

Q. 2. Distinguish between first and second order differential equation ?

Q. 3. When a differential equation is called a homogeneous equation ?

Q. 4. Solve the following differential equations

(i) $(1 + x)\ ydx + (1 - y)\ (xdy) = 0$

(ii) $(x^2 - yx^2)dy - (y^2 + xy^2)\ dx = 0$

(iii) $(a - x)\ dy - (a + y)\ dx = 0$

(iv) $\frac{dy}{dx} = e^x - y + x^2\ c^{-y}$

11.9 SUGGESTED READINGS

Aggarwal, C.S & R.C Joshi : Mathematics for students of Economics (New Academic Publishing Co.)

Black, J & J.F Bradley : Essential Mathematics for Economics (John Willey and Sons)

Chander, R : Lectures on Elementary Mathematic (New Academic Publishing Co.)

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M.A. ECONOMICS		LESSON NO. 12
COURSE NO. 103	SEMESTER IST	UNIT–III

DIFFERENCE EQUATIONS

Structure

12.1 INTRODUCITON

Many problems in probability give rise to difference equations. Difference equations relate to differential equations as discrete mathematics. Anyone who has made a study of differential equations will know that even supposedly elementary examples can be hard to solve. By contrast, elementary difference equations are relatively easy to deal with.

Aside from probability, computer scientists take an interest in difference equations for a number of reasons e.g., difference equations frequently arise when determining the cost of an algorithm in big–Onotation. Since difference equations are readly handled by program, a standard approach to solving a nasty differential equation is to convert it to an approximately equivalent difference equation.

12.2 OBJECTIVES

After reading this unit you should be able to :

- Difference Equations and their concepts.
- First - Order and Higher - Order Difference Equation.

12.3 DIFFERENCE EQUATION

An ordinary difference equation is an equation showing the relation between the independence variable x, the dependent variable y_x and its finite difference $\Delta yx, \Delta^2 yx, \Delta^3 yx$.

The order of the difference equation is the order of the highest difference contained in the equation.

In the form, $(y_x, y_{x-1}, y_{x-2}, ..., y_{x-n})$ 0

This form is known as the Lagged form

General form :

Suppose that we are seeking the solution to the first order difference equation.

$$y_{t+1} + a_{yt} = c \quad ...(1)$$

where a and c are two constants.

∴ General solution will comprise of

Particular solution (yp) and complementary solution (yc)

For complementary solution, $y_{t+1} + a_{yt} = 0$

Put, $yt = AB^t$, and $y_{t+1} = Ab^{t+1}$,

∴ Putting in equation (2), we get

$$Ab^{t+1} + aAb^t = 0$$

$$\Rightarrow Ab^t . b + aAb^t = 0$$

$\Rightarrow$ $b + a = 0$ {because Ab^t is common and when it divided with zero gives answer 0}

$\Rightarrow$ $b = -a$

$\Rightarrow$ the $yc = Ab^t = A(-a)^t$

$\Rightarrow$ $\boxed{yc = A(-a)^t}$

Now, the particular solution.

Put $yt = k$, and also, $y_{t+1} = k$

$\Rightarrow K + ak = C$

and $K = \dfrac{C}{1+a}$

$\Rightarrow$ Particular sol. will be

$$yp \ (= k) = \frac{c}{1+a}(a^1 - 1)$$

In case $a = -1$, then, the particular solution $\dfrac{c}{1+a}$ is not defined, then,

Put $y_t = Kt$ and $y_{t+1} = k(t+1)$

now substitution it in equation (2) we get

$$K(t+1) + a\,(kt) = c$$

and $kt + k + akt = c$

$$k(t + 1 + at) = c$$

$$\Rightarrow k = \frac{c}{t + 1 + at} = ct$$

$\Rightarrow$ General solution

$$yt = A(-a)^t + \frac{C}{1+a} \quad [\text{Case of } ^1 \quad a - 1]$$

$$yt = A(-a)^t + \frac{C}{1+t+at} \quad [\text{Case of } a = -1]$$

12.3.1 FIRST ORDER DIFFERENCE EQUATION

$\boldsymbol{y_{t+1} - 5_{yt} = 1}$, **and** $\boldsymbol{y_0 = 7/4}$

Sol. The given equation

$$y_{t+1} - 5y_t = 1$$

For, complementary solution.

$$yt = Ab^t,\ y_{t+1} = Ab^{t+1}$$

 Ab^{t+1} - 5Ab^t = 0

$$\Rightarrow Ab^{t+1} - 5Ab^t = 0$$

$$\Rightarrow Ab^t.b^1 - 5Ab^t = 0$$

$$\Rightarrow Ab^t(b - 5) = 0$$

$$\Rightarrow b - 5 = 0$$

$$\Rightarrow b = 5$$

$$\Rightarrow yc = A(5)^t$$

For particular solution

Put $yt = k = y_{t+1}$

$K - 5k = 1$

$-4k = 1$

$k = -1/4$

$$\boxed{yp = -1/4}$$

General solution is,

$$yt = yc + yp$$

$yt \;= A(5)^t - 1/4$

Given conditions, $y(0) = 7/4$

$y(t) = A(5)^0 - 1/4$

$7/4 = A \times 1 - 1/4$

$$\frac{7}{4} + \frac{1}{4} = A$$

$$\frac{8}{4} = A$$

$$\boxed{A = 2}$$

i.e. $y(t) = 2(5)^t - \dfrac{1}{4}$

$$y(t) = 2(5)^t - \frac{1}{4}$$

Solve it :

Q. $y_{t+1} + 3y_t = 4, y_0 = 4$

Q. $2y_{t-1} - yt = 6 \; (y_0 = 7)$

Q. $y_{t+1} = 0.2 \; y_t + 4 \; (y_0 = 4)$

Note that $\times y_t = y_{t+1} - y_t,$

Q. $\times y_t = \times 2y_t - 9$

Q. $2y_{x+1} = 6y_x - 4, y_0 = 2$

Ans. $3^x + 1.$

Q. $y_{t+1} = y_t^{+2}, y_0 = 2,$

Ans. $y = 2 \; (1)^t + 2t$

Q. $\Delta yx = - 7yx$

Ans. $y_n = A(-6)^x.$

Dynamic Stability of b

The time path of b^t ($b \neq 0$) will be

Non-oscillatory } if $\begin{cases} b > 0 \\ b < 0 \end{cases}$

Divergent if $\left\{ \begin{array}{l} |b| > 1 \\ |b| < 1 \end{array} \right\}$

Convergent

12.4 HIGHER ORDER DIFFERENCE EQUATION

The second difference of *yt* is transformable into a sum of terms involving a two-period time lag. Similarly, a third order difference equation is one that involves a three-period time lag etc.

A simple variety of second-order difference equation that takes the form.

$$y_{t+2} + a_1 y_{t-1} + a_2 y_t = c.$$

Particular Solution *yp*

Put, $y_t = k, \; y_{t+1} = k, \; y_{t+2} = k$

 (*i.e.*) Putting it in (1), we get

$$k + a_1 k + a_2 k = c$$

and $k(1 + a_1 + a_2 = c$

$$\therefore \quad k = \frac{c}{1 + a_1 + a_2} [a_1 + a_2 = - 1]$$

In case $a_1 + a_2 = -1$

then, put $y_t = kt, \; y_{t+1} = k(t+1), \; y_{t+2} = k(t + 2)$

$\therefore$ $k(t + 2) + a_1 k\,(t + 1) + a_2 kt = c$

and $k = \dfrac{c}{(1 + a_1 + a_2)t + a_1 + 2} = \dfrac{c}{a_1 + 2}$

 The particular solution, is

$$yp = \frac{c}{a_1 + 2} t$$

Complementary function

Put, $y_t = Ab^t$,

$y_{t+1} = b.Ab^t$

$y_{t+2} = b^2.Ab^t$

 Trail solution of the equation

$b^2 AB^t + a_1 bAb^t + a_2 Ab^t = 0$

  $b^2 + a_1 b + a_2 = 0$

This quadratic equation possesses two characteristc roots

$$b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Case I. (distinct real roots)

(*i.e.*) $(b_1 \pm b_2)$

 $yc = A_1 b_1^t + A_2 b_2^t$

Note. In case of stability and converge both $(b_1 = b_2)$ $|b_1| < 1, |b_2| < 1$

Case II. (repeated real roots)

then, $yc = A_3 b^t + A_4 t b^t$

Note. In case of stability and converges when $|b|<1$

Case III. (Complex roots)

$b_1, b_2 = h \pm vi$

 $yc = R^t$ [As cos 6]39t + A_6 .sin 6]39t]

$$R = \sqrt{h^2 + v^2}$$

Note. In case of stability $|h| < 1$

Q. Find the solution of $y_{t+2} + y_{t+1} - 2y_t = 12$, $y_0 = 4$, $y_1 = 5$

Sol. The given equation is

$$y_{t+2} + y_{t+1} - 2_{yt} = 12 \ldots(1)$$

 For complementary solution

$$y_t = AB^t,\ y_{t+1} = Ab^t,\ y_{t+2} = b^2.Ab^t$$

 Putting in (1) we get

$$b^2Ab^t + bab^t - 2Ab^t = 0$$

 $$b^2 + b - 2 = 0$$

By solving this quadratic equation, we have following roots

$$b_1 = 1, \text{ and } b_2 = 2$$

 Complementary solution is

$$yc = A_1(1)^t + A_2\ (-2)^t$$

For particular solution. Put

$$y_t = k, \quad y_{t+1} = k, \quad y_{t+2} = k$$

$$k + k - 2k = 12$$

$$0.k = 12$$

Now,

$$yp = \frac{12t}{a_1 + 2} = \frac{12t}{1+2} = \frac{12t}{3}$$

$$yp = 4t$$

 General solution

$$y(t) = A_1(1)^t + A_2\ (-2)^t + 4t$$

Putting $y(0) = 4$, and $y(1) = 5$, we get

$$y_0 = 4 = A_1 + A_2$$

$$y_1 = 5\ A_1 - 2\ A_2 + 4$$

Solving, this we find, that

$$A_1 = 3,\ A_2 = 1$$

□ definite sol. will be,

$$y(t) = 3(1)^t + (-2)^t + 4t$$

Q. Find the solution of $y_{t+2} + 6y_{t+1} + 9yt = 4$

Sol. The given equation is

$$y_{t+2} + 6_{yt+1} + 9_{yt} = 4$$

□ For complementary solution,

$$y_t = Ab^t,\ y_{t+1} = Ab^t.b + y_{t+1} = Ab^t.b^2$$

□ Putting it in (1), we get

$$b^2 + 6b + 9 = 0$$

$$□\ b_1 = -3,\ b_2 = -3$$

$$□\ y_c = A_3\ (-3)^t + A_4\ t(-3)^t$$

For particular solution.

Put $y_{t+2} = k,\ y_{t+1} = k,\ y_t = k$

$$□\ k + 6k + 9k = 4$$

$$16\ k = 4$$

$$k = 4/16$$

$$k = 1/4$$

$$□\quad yp = 1/4$$

Hence, general solution is

$$y(t) = A_3(-3)^t + A_4{}^t + (-3)^t + \frac{1}{4}$$

Q. Find the general solution of

$$y_{t+2} - 4y_{t+1} + 16y = 0$$

Sol. The given equation is

$$y_{t+2} - 4_{yt+1} + 16y = 0$$

For complementary solution.

$$y_t = Ab^t,\ y_{t+1} = b.Ab^t\ ,\ y_{t+2} = b^2.Ab^t$$

$$b^2\ Ab^1 - 4bAb^t + 16Ab^t = 0$$

 $b^2 - 4b + 16 = 0$

 $b_1\ ,\ b^2 = \dfrac{4 \pm \sqrt{16 - 4 ´ 1 ´ 16}}{2 ´ 1}$

$$= \frac{4 \pm \sqrt{16 - 64}}{2}$$

$$= \frac{4 \pm \sqrt{-48}}{2}$$

$$= \frac{4 \pm \sqrt{(-1)(4)(12)}}{2}$$

$$= \frac{4 \pm 2\sqrt{12}i}{2}$$

$$= 2 \pm \sqrt{12}i$$

 R = 4

 $\cos 6\overline{)39} = \dfrac{h}{R} = \dfrac{2}{4} = \dfrac{1}{2} \Rightarrow \sin q = \dfrac{u}{R} = \dfrac{\sqrt{12}}{4} = \dfrac{2\sqrt{3}}{2} = \dfrac{\sqrt{3}}{2}$

$\therefore\ yc = (4)^t \left(A_5 \cos \pi/2t + A_6 . \sin \frac{\sqrt{3}}{2} . \pi t \right)$

$\therefore\ yp = 0$

Hence, total solution

$$yt = 4t \left(A_5 \cos \pi/2t + A_6 \sin \frac{\sqrt{3}}{2} \pi t \right)$$

12.5 SUMMARY

We end this lesson by summarizing what we have covered in it.

i) Difference equations and their related concepts.

ii) First-Order and Higher-Order Difference Equation.

12.6 LESSON END EXERCISE

Q. Solve the following difference equations.

$y_{t+2} + 3y_{t+1} - 7/4yt = 9,\ (y_0 = 6;\ y_1 = 3)$

Sol.

Q. $y_{t+2} - 2y_{t+1} + 2yt = 1\ (y_0 = 3;\ y_1 = 4)$

Q. $y_{x+2} - 6y_{x+1} + 8y_x = 0$

Q. $y_{x+2} - 3y_{x+1} + 5y_x = 0$

Q. $y_{t+2} + y_{t+1} + y_{t} = 0$

Q. $y_{t+2} - 4y_t = 0$, given $y_0 = -2,\ y_1 = 16$

Q. $4y_{x+2} - 12y_{x+1} + 9y_x = 0$, given $y_0 = 2$, $y_1 = -3$.

Sol.

Q. Solve, $y_x - 2y_{x-1} = 4$, given $y_1 = 4$.

Q. Solve, $yx - 2yx - 1 + 4 = 0$, given $y_0 = 5$

Q. Solve $y_x - 6y_{x-1} + 8y_{x-2} = 5$

12.7 SUGGESTED READINGS

Aggarwal, C.S & R.C Joshi : Mathematics for students of Economics (New Academic Publishing Co.)

Black, J & J.F Bradley : Essential Mathematics for Economics (John Willey and Sons)

Chander, R : Lectures on Elementary Mathematic (New Academic Publishing Co.)

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M.A. ECONOMICS		**LESSON NO. 13**
COURSE NO. 103	**SEMESTER IST**	**UNIT–III**

SIMULTANEOUS DIFFERENTIAL EQUATIONS

Structure

13.1 INTRODUCTION

Given a system of linear autonomous differential equations, the intertemporal equilibrium level will be asymptotically stable. *i.e.*, $y(t)$ will converge to $\vec{y}$ as t ® ¥ , it and only if all the characteristic roots are negative. In the case of complex roots, the real part must be negative. If all the roots are positive, the system will be unstable. A saddle-point equilibrium, in which roots assume different signs, will generally be unstable.

If, however, the initial conditions for y_1 and y_2 satisfy the condition.

$$y_2 = \left(\frac{r_1 - a_{11}}{a_{12}}\right)(y_1 - \bar{y}_1) + \bar{y}_2$$

where r_1 = the negative root, we have what is called a saddle path, and $y_1(t)$ and $y_2(t)$ will converge to their intertemporal equilibrium level (see Example 10).

A phase diagram for a system of two differential equations, linear or nonlinear, graphs y_2 on the vertical axis and y_1 on the horizontal axis The y_1, y_2 plane is called the phase plane. Construction of a phase diagram is easiest explained in terms of an example.

13.2 OBJECTIVES

After reading this unit you should be able to -

- Simultaneous Differential Equations.
- Simultaneous Difference Equations.
- Phase Diagrams.

13.3 SIMULTANEOUS DIFFERENTIAL EQUATIONS

Solve the following system of first-order, autonomous, linear differential equations.

$$\dot{y}_1 = -8y_1 + 5y_2 + 4 \qquad y_1(0) = 7$$

$$\dot{y}_2 = 3.25\,y_1 - 4y_2 + 22 \qquad y_2(0) = 21.5$$

1. Putting them in matrix form for ease of computation,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -8 & 5 \\ 3.25 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 22 \end{bmatrix}$$

$$\dot{\mathbf{Y}} = \mathbf{AY} + \mathbf{B}$$

2. Then find the complementary functions. Assuming distinct real roots.

$$y_c = k_1 C_1 e^{r_1 t} + k_2 C_2 e^{r_2 t}$$

$$\text{where } r_1, r_2 = \frac{\text{Tr(A)} \pm \sqrt{[\text{Tr(A)}]^2 - 4|\text{A}|}}{2}$$

$$\text{T}r\,(\text{A}) = -12, \text{ and } |\text{A}| = 15.75.$$

$$r_1, r_2 = \frac{-12 \pm \sqrt{(12)^2 - 4(15.75)}}{2} = \frac{-12 \pm 9}{2}$$

$$r_1 = -1.5 \qquad r_2 = -10.5$$

3. Next we find the eigenvectors C_i from

$$(\text{A} - r_i\text{I})\ \text{C}_i = \begin{bmatrix} a_{11} - r_i & a_{12} \\ a_{21} & a_{22} - r_i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

(a) For $r_1 = -1.5$,

$$\begin{bmatrix} -8-(1.5) & 5 \\ 3.25 & -4-(-1.5) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -6.5 & 5 \\ 3.25 & -2.5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

By simple multiplication of row by column,

$$-6.5c_1 + 5c_2 = 0 \qquad c_2 = 1.3\ c_1$$

$$3.25c_1 - 2.5c_2 = 0 \qquad c_2 = 1.3\ c_1$$

If $c_1 = 1$, then $c_2 = 1.3$. Thus, the eigenvector C_i coresponding to $r_1 = -1.5$ is

$$\text{C}_1 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.3 \end{bmatrix}$$

and the first elements of the complmentary function of the general solution are

$$y_c^1 = k_1 \begin{bmatrix} 1 \\ 1.3 \end{bmatrix} e^{-1.5t} = \begin{bmatrix} k_1 e^{-1.5t} \\ 1.3k_1 e^{-1.5t} \end{bmatrix}$$

(b) Substituting next for $r_2 = -10.5$

$$\begin{bmatrix} -8-(-10.5) & 5 \\ 3.25 & -4-(-10.5) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2.5 & 5 \\ 3.25 & 6.5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

Then simply multiplying the first row by the column, since the final results will always be the same due to the singularity of the $(A - r_i I)$ matrix

$$2.5c_1 + 5c_2 = 0 \qquad c_1 = -2c_2$$

If $c_2 = 1$, then $c_1 = -2$; the eigenvector C_2 for $r_2 = -10.5$ is

$$C_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and the second elements of the general complementary function are

$$y_c^2 = k_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-10.5t} = \begin{bmatrix} -2k_2 e^{-10.5t} \\ k_2 e^{-10.5t} \end{bmatrix}$$

This makes the complete complementary solution,

$$y_1(t) = k_1 e^{-1.5t} - 2k_2 e^{-10.5t}$$

$$y_2(t) = 1.3k_1 e^{-1.5t} + k_2 e^{-10.5t}$$

4. Now find the intertemporal equilibrium solutions for y_p,

$$y_p = \overline{Y} = -\mathbf{A^{-1}\ B}$$

where $A = \begin{bmatrix} -8 & 5 \\ 3.25 & -4 \end{bmatrix}$, $C = \begin{bmatrix} -4 & -3.25 \\ -5 & -8 \end{bmatrix}$,

Adj. $A = \begin{bmatrix} -4 & -5 \\ -3.25 & -8 \end{bmatrix}$, $A^{-1} = \dfrac{1}{15.75} \begin{bmatrix} -4 & -5 \\ -3.25 & -8 \end{bmatrix}$

Substituting above, $\overline{Y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = -\dfrac{1}{15.75} \begin{bmatrix} -4 & -5 \\ -3.25 & -8 \end{bmatrix} \begin{bmatrix} 4 \\ 22 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$

Thus the complete general solution, $y(t) = y_c + y_p$, is

$$y_1(t) = k_1 e^{-1.5t} - 2k_2 e^{-10.5t} + 8$$

$$y_2(t) = 1.3k_1 e^{-1.5t} + k_2 e^{-10.5t} + 12 \qquad (19.27)$$

To find the definite solution, simply evaluate the above equations at $t = 0$ and use the initial conditions.

(0) = 7, y_2 (0) = 21.5

$$y_1(0) = k_1 - 2k_2 + 8 = 7$$

$$y_2(0) = 1.3\, k_1 + k_2 + 12 = 21.5$$

Solved simultaneously, $k_1 = 5$ $\quad k_2 = 3$

Substituting in (19.27),

$$y_1(t) = 5e^{-1.5t} - 6e^{-10.5t} + 8$$

$$y_2(t) = 6.5e^{-1.5t} + 3e^{-105t} + 12$$

With $r_1 = -1.5 < 0$, $r_2 = -10.5 < 0$, the equilibrium is dynamically stable.

Solve the following system of differential equations,

$$\dot{y}_1 = 2y_2 - 6 \qquad y_1(0) = 1$$

$$\dot{y}_2 = 8y_1 - 16 \qquad y_2(0) = 4$$

1. In matrix form, $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -6 \\ -16 \end{bmatrix}$

$$\dot{\mathbf{Y}} = \mathbf{AY} + \mathbf{B}$$

2. Finding the characteristic roots,

$$r_1, r_2 = \frac{\text{Tr(A)} \pm \sqrt{[\text{Tr(A)}]^2 - 4|\text{A}|}}{2}$$

$$r_1, r_2 = \frac{0 \pm \sqrt{(0)^2 - 4(-16)}}{2} = \frac{\pm 8}{2}$$

$$r_1 = -4 \qquad r_2 = 4$$

3. Now determine the eigenvectors.

For $r_1 = -4$.

$$\begin{bmatrix} 0-(-4) & 2 \\ 8 & 0-(-4) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$4c_1 + 2c_2 = 0 \qquad c_2 = -2c_1$$

If $c_1 = 1$, then $c_2 = -2$, and the first elements of the complementary function are

$$y_c^1 = k_1\begin{bmatrix}1\\-2\end{bmatrix}e^{-4t} = \begin{bmatrix}k_1e^{-4t}\\-2k_ie^{-4t}\end{bmatrix}$$

For $\qquad r_1 = 4, \begin{bmatrix}0-4 & 2\\8 & 0-4\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-4 & 2\\8 & -4\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = 0$

$$-4c_1 + 2c_2 = 0 \qquad c_2 = 2c_1$$

If $c_1 = 1$, then $c_2 = 2$, and the second elements of the complementary function are

$$y_2^2 = k_2\begin{bmatrix}1\\2\end{bmatrix}e^{4t} = \begin{bmatrix}k_2e^{4t}\\2k_2e^{4t}\end{bmatrix}$$

This means the general complementary functions are

$$y_t(t) = k_1e^{-4t} + k_2e^{4t}$$

$$y_2(t) = -2k_1e^{-4t} + 2k_2e^{4t}$$

4. For the steady-state solutions y_p,

$$y_p = \bar{Y} = -A^{-1}B$$

$$\bar{Y} = \begin{bmatrix}\bar{y}_1\\\bar{y}_2\end{bmatrix} = -\frac{1}{-16}\begin{bmatrix}0 & -2\\-8 & 0\end{bmatrix}\begin{bmatrix}-6\\-16\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

Thus the complete general solution, $y(t) = y_c + y_p$, is

$$y_1(t) = k_1e^{-4t} + k_2e^{4t} + 2$$

$$y_2(t) = -2k_1e^{-4t} + 2k_2e^{4t} + 3$$

5. We then find the definite solution from the initial conditions, $y_1(0) = 1$, $y_2(0) = 4$.

$$y_1(0) = k_1 + k_2 + 2 = 1$$

$$y_2(0) = -2k_1 + 2k_2 + 3 = 4$$

Solved simultaneously, $k_1 = -0.75 \quad k_2 = -0.25$

Substituting in (19.28), we have the final solution,

$$y_1(t) = 0.75e^{-4t} - 0.25\ e^{4t} + 2$$

$$y_2(t) = 1.5\ e^{-4t} + 0.5\ k_2\ e^{4t} + 3 \qquad (19.28)$$

With $r_1 = -4 < 0$ and $r_2 = 4 > 0$, we have a saddle-point solution. Saddle-point solutions are generally unstable unless the initial conditions fall on the saddle path :

$$y_2 = \left(\frac{r_1 - a_{11}}{a_{12}}\right)(y_1 - \bar{y}_1) + \bar{y}_2$$

Substituting, $$y_2 = \left(\frac{-4 - 0}{2}\right)(y_1 - 2) + 3$$

$$y_2 = 7 - 2y_1$$

This is the equation for the saddle path, which was graphed in Fig. 19.2 of Example 10. Substituting the initial conditions, $y_1(0) = 1$, $y_2(0) = 4$, we see

$$4 \neq 7 - 2(1) = 5$$

Since the initial conditions do not fall on the saddle path, the system is unstable.

Solve the following system of equations.

$$\dot{y}_1 = 4y_1 + 7y_2 + 3 \qquad y_1(0) = 7$$

$$\dot{y}_2 = y_1 - 2y_2 + 4 \qquad y_2(0) = 10$$

1. Converting to matrices,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\mathbf{\dot{Y} = AY + B}$$

2. The characteristic roots are

$$r_1, r_2 = \frac{2 \pm \sqrt{(2)^2 - 4(-15)}}{2} = \frac{2 \pm 8}{2}$$

$$r_1 = -3 \qquad r_1 = 5$$

The eigenvector for $r_1 = -3$,

$$\begin{bmatrix} 4-(-3) & 7 \\ 1 & -2-(-3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$7c_1 = 7c_2 = 0 \qquad c_1 = -c_2$$

If $c_2 = 1$, then $c_2 = -1$, and

$$y_c^1 = k_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} -k_1 e^{-3t} \\ k_1 e^{-3t} \end{bmatrix}$$

For $r_2 = 5$,

$$\begin{bmatrix} 4-5 & 7 \\ 1 & -2-5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-c_1 + 7c_2 = 0 \qquad c_1 = 7c_2$$

If $c_2 = 1$, then $c_1 = 7$, and

$$y_c^2 = k_2 \begin{bmatrix} 7 \\ 1 \end{bmatrix} e^{5t} = \begin{bmatrix} 7k_2 e^{5t} \\ k_2 e^{5t} \end{bmatrix}$$

The general complementary functions are

$$y_1(t) = -k_1 e^{-3t} + 7k_2 e^{5t}$$

$$y_2(t) = -k_1 e^{-3t} + k_2 e^{5t}$$

The steady-state solutions y_p are

$$y_p = \bar{Y} = -A^{-1} B$$

$$\bar{Y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = -\frac{1}{-15} \begin{bmatrix} -2 & -7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 31 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$$

and the complete general solution is

$$y_1(t) = -\,k_1e^{-3t} + 7k_2\,e^{5t} - 6 \tag{19.29}$$

$$y_2(t) = k_1e^{-3t} + k_2e^{5t} \; - 1$$

The definite solution, given $y_1(0) = 7$, $y_2(0) = 10$, is

$$y_1(0) = -\,k_1 + 7k_2 - 6 = 7$$

$$y_2(0) = k_1 + k_2 - 1 = 10$$

$$k_1 = 8 \qquad\qquad k_2 = 3$$

Substituting in (19.29) for the final solution.

$$y_1(t) = -\,8e^{-3t} + 21e^{5t} - 6$$

$$y_2(t) = -\,8e^{-3t} + 3e^{5t} - 1$$

With $r_1 = -3 < 0$ and $r_2 = 5 > 0$, we again have a saddle-point solution which will be unstable unless the initial conditions fulfill the saddle-path equation :

$$y_2 = \left(\frac{r_1 - a_{11}}{a_{12}}\right)(y_1 - \bar{y}_1) + \bar{y}_2$$

Substituting $$\dot{y}_2 = \left(\frac{-3-4}{7}\right)[y_1 - (-6)] + (-1)$$

$$\dot{y}_2 = -7 - \dot{y}_1$$

Employing the initial conditions, $y_1(0) = 7$, $y_2(0) = 10$.

$$10 \neq -7 - (7) = -14$$

Since the initial conditions do not satisfy the saddle path equation, the system of equations is unstable.

Solve the following system of nonlinear, autonomous, first-order differential equations in which one or more derivative is a function of another derivative.

$$\dot{y}_1 = 4y_1 + y_2 + 6 \qquad\qquad y_1(0) = 9$$

$$\dot{y}_2 = 8y_1 + 5\dot{y}_2 - \dot{y}_1 - 6 \qquad y_2(0) = 10$$

1. Rearranging the equations to conform with (19.7) and setting them in matrix form.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$A_1\dot{Y} = A_2Y + B$$

2. Find the characteristic roots from the characteristic equation,

$$|A_2 - r_i A_1| = 0$$

where, dropping the i subscript for simplicity,

$$|A_2 - rA_1| = \begin{bmatrix} 4 & 1 \\ 8 & 5 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4-r & 1 \\ 8-r & 5-r \end{bmatrix} = 0$$

$$r^2 - 8r + 12 = 0$$

$$r_1 = 2 \qquad r_2 = 6$$

3. Find the eigenvectors C_i where

$$(A_2 - r_i A_1)C_i = 0$$

and $$(A_2 - r_i A_1)C_i = \begin{bmatrix} 4-r_i & 1 \\ 8-r_i & 5-r_i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Substituting for $r_1 = 2$,

$$\begin{bmatrix} 4-2 & 1 \\ 8-2 & 5-2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$2c_1 + c_2 = 0 \qquad c_2 = -2c_1$$

If $c_1 = 1, c_2 = -2$, and

$$y_c^1 = k_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t} = \begin{bmatrix} k_1 \\ -2k_1 \end{bmatrix} e^{2t}$$

Now substituting for $r_2 = 6$

$$\begin{bmatrix} 4-6 & 1 \\ 8-6 & 5-6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-2c_1 + c_2 = 0 \qquad c_2 = 2c_1$$

If $c_1 = 1, \quad c_2 = 2$, and

$$y_c^2 = k_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{6t} = \begin{bmatrix} k_2 \\ 2k_2 \end{bmatrix} e^{6t}$$

Adding the two components of the complementary functions,

$$y_1(t) = k_1 e^{2t} + k_2 e^{6t}$$

$$y_2(t) = 2k_1 e^{2t} + 2k_2 e^{6t}$$

4. For the particular integral y_p,

$$\vec{Y} = -A_2^{-1} B$$

where $B = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 1 \\ 8 & 5 \end{bmatrix}$, $|A_2| = 20 - 8 = 12$, and A_2^{-1}

$$= \frac{1}{12} \begin{bmatrix} 5 & -1 \\ -8 & 4 \end{bmatrix}.$$

Substituting,

$$\vec{Y} = \begin{bmatrix} \vec{y}_1 \\ \vec{y}_2 \end{bmatrix} = -\frac{1}{12} \begin{bmatrix} 5 & -1 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

Adding the particular integrals to the complementary functions,

$$y_1(t) = k_1 e^{2t} + k_2 e^{6t} - 3$$

$$y_2(t) = -\ 2k_1 e^{2t} + 2k_2 e^{6t} + 6 \qquad (19.30)$$

For the definite solution, set t = 0 in (19.30) and use $y_1(0) = 9$. $y_2(0) = 10$

$$y_1(0) = k_1 + k_2 - 3 = 9$$

$$y_2(0) = -2k_1 + 2k_2 + 6 = 10$$

$$k_1 = 5 \qquad k_2 = 7$$

Then substituting back in (19.30),

$$y_1(t) = 5e^{2t} + 7e^{6t} - 3$$

$$y_2(t) = -10e^{2t} + 14e^{6t} + 6$$

With $r_1 = 2 > 0$ and $r_2 = 6 > 0$, the system of equations will be dynamically unstable.

Solve the following system of differential equations.

$$\dot{y}_1 = -y_1 + 4y_2 - 0.5\dot{y}_2 - 1 \qquad y_1(0) = 4.5$$

$$\dot{y}_2 = 4y_1 - 2y_2 - 10 \qquad y_2(0) = 16$$

Rearranging and setting in matrix form,

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -10 \end{bmatrix}$$

$$A_1 Y = A_2 Y + B$$

From the characteristic equation,

$$|A_2 - rA_1| = \begin{bmatrix} -1-r & 4-0.5r \\ 4 & -2-r \end{bmatrix} = 0$$

we find the characteristic roots,

$$r^2 + 5r - 14 = 0$$

$$r_1 = -7 \qquad r_2 = 2$$

We next find the eigenvectors C_i from the eigenvalue problem,

$$(A_2 - r_1 A_1)C_i = \begin{bmatrix} -1-r & 4-0.5r \\ 4 & -2-r \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

Substituting for $r_1 = -7$,

$$\begin{bmatrix} -1-(-7) & 4-0.5(-7) \\ 4 & -2-(-7) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 & 7.5 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$6c_1 + 7.5c_2 = 0 \qquad c_1 = -1.25c_2$$

If $c_2 = 1, c_1 = -1.25$, and

$$y_c^1 = k_1\begin{bmatrix} -12.5 \\ 1 \end{bmatrix} e^{-7t} = \begin{bmatrix} -1.25k_1e^{-7t} \\ k_1e^{-7t} \end{bmatrix}$$

Substituting for $r_1 = 2$,

$$\begin{bmatrix} -1-2 & 4-0.5(2) \\ 4 & -2-2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-3c_1 + 3c_2 = 0 \qquad c_1 = c_2$$

If $c_2 = 1, c_1 = 1$, and

$$y_c^2 = k_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} k_2e^{2t} \\ k_2e^{2t} \end{bmatrix}$$

This makes the complete general complementary functions,

$$y_1(t) = -1.25\ k_1e^{-7t} + k_2e^{2t}$$

$$y_2(t) = -\ k_1e^{-7t} + k_2e^{2t}$$

Finding the particular integral $\vec{Y} = -A_2^{-1}B$.

where $A_2 = \begin{bmatrix} -1 & 4 \\ 4 & -2 \end{bmatrix}$, $A_2^{-1} = \dfrac{1}{-14}\begin{bmatrix} -2 & -4 \\ -4 & -1 \end{bmatrix}$ and

$$\bar{Y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -2 & -4 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ -10 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

By adding the particular integrals to the complementary functions, we get

$$y_1(t) = -1.25k_1e^{-7t} + k_2e^{2t} + 3$$

$$y_2(t) = k_1e^{-2t} + k_2e^{2t} + 1$$

For the definite solution, we set $t = 0$ in (19.31) and use $y_1(0) = 4.5$, $y_2(0) = 16$.

$$y_1(0) = -1.25k_1 + k_2 + 3 = 4.5$$

$$y_2(0) = k_1 + k_1 + 1 = 16$$

$$k_1 = 6 \qquad k_2 = 9$$

Finally, substituting in (19.31),

$$y_1(t) = -7.5e^{-7t} + 9e^{2t} + 3$$

$$y_2(t) = -6e^{-7t} + 9e^{2t} + 1$$

With characteristic roots of different signs, we have a saddle-point equilibrium which will be unstable unless the initial conditions happen to coincide with a point on the saddle path.

Solve the following

$$\dot{y}_1 = -3y_1 \square 4y_2 \square 0.5\dot{y}_2 \square 1 \qquad y_1(0) - 22.2$$

$$\dot{y}_2 = -2y_1 \square 4y_2 \square \dot{y}_1 \square 10 \qquad y_2(0) - 3.9$$

In matrix form,

$$\begin{bmatrix} 1 & 0.5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$A_1Y = A_2Y + B$$

2. The characteristic equation is

$$|A_2 - rA_1| = \begin{vmatrix} -3-r & -1-0.5r \\ -2-r & -4-r \end{vmatrix} = 0$$

$$0.5r^2 + 5r + 10 = 0$$

Multiplying by 2 and using the quadratic formula, the characteristic roots are

$r_1 = -7.235 \qquad r_2 = -7.265$

3. The eigenvector for $r_1 = -7.235$ is

$$\begin{bmatrix} -3-(-7.235) & -1-0.5(-7.235) \\ -2-(-7.235) & -4-(-7.235) \end{bmatrix} \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 4.235 & 2.6175 \\ 5.235 & 3.235 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$4.235c_1 + 2.6175c_2 = 0 \qquad c_2 = -1.62c_1$

If $c_1 = 1. c_2 = -1.62$, and

$$y_c^1 = k_1 \begin{bmatrix} 1 \\ -1.62 \end{bmatrix} e^{-7.235t} = \begin{bmatrix} k_1 e^{-7.235t} \\ -1.62k_1 e^{-7.235t} \end{bmatrix}$$

For $r_2 = 2 - 2.765$,

$$\begin{bmatrix} -3-(-2.765) & -1-0.5(-2.765) \\ -2-(-2.765) & -4-(-2.765) \end{bmatrix} \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} -0.235 & 0.3825 \\ 0.765 & -1.235 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$-0.235c_1 + 0.3825c_1 = 0 \qquad c_1 = 1.62c_2$

If $c_2 = 1, c_1 = 1.62$ and

$$y_c^2 = k_2 \begin{bmatrix} 1.62 \\ 1 \end{bmatrix} e^{-2.765t} = \begin{bmatrix} 1.62k_2 e^{-2.765t} \\ k_2 e^{-2.765t} \end{bmatrix}$$

The complete complementary function, then is

$$y_1(t) = k_1 e^{+2.45t} + 1.65k_2 e^{2.765t}$$

$$y_2(t) = -1.65k_1 e^{7.235t} + k_2 e^{-2.7651}$$

4. The particular integral $\vec{Y} = -A_2^{-1}B$ is

$$\vec{Y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution is

$$y_1(t) = k_t e^{-7.235t} + 1.62k_2 e^{-2.765t} + 1$$

$$y_2(t) = 1.62k_1 e^{-7.235t} + k_2 e^{-2.765t} + 2 \qquad (19.32)$$

5. Using $y_1(0) = 22.2$ $y_2(0) = 3.9$ to solve for k_1 and k_2

$$k_1 + 1.62k_2 + 1 = 22.2$$

$$-1.62k_1 + k_2 + 2 = 3.9$$

$$k_1 = 5 \qquad k_2 = 10$$

Substituting in (19.32) for the definite solution,

$$y_1(t) = 5e^{-7.235t} + 16.2e^{-2.765t} + 1$$

$$y_2(t) = -8.1e^{-7235t} + 10e^{-2.765t} + 2$$

With both characteristic roots negative, the intertemporal equilibrium is stable.

13.4 SIMULTANEOUS DIFFERENCE EQUATIONS

Solve the following system of first-order linear difference equations in which no difference is a function of another difference

$$x_t = 0.4x_{t-1} + 0.6y_{t-1} + 6 \qquad x_0 = 14$$

$$y_t = 0.1x_{t-1} + 0.3y_{t-1} + 5 \qquad y_0 = 23$$

1. Setting them in matrix form,

$$\begin{bmatrix} x_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.1 & 0.3 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$Y_t = AY_{t-1} + B$$

2. Using (19.3) on the characteristic equation $|A - r_t I| = 0$, find the characteristic roots.

$$r_1, r_2 = \frac{0.7 \pm \sqrt{(0.7)^2 - 4(0.06)}}{2} = \frac{0.7 \pm 0.5}{2}$$

$$r_1 = 0.6 \qquad r_2 = 0.1$$

3. The eigenvector for $r_1 = 0.6$ is

$$\begin{bmatrix} 0.4-0.6 & 0.6 \\ 0.1 & 0.3-0.6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.6 \\ 0.1 & -0.3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$-0.2c_1 + 0.6c_2 = 0 \qquad c_1 = 3c_2$

If $c_2 = 1$, $\quad c_1 = 3$, and we have

$$k_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}(0.6)^t = \begin{bmatrix} 3k_1(0.6)^t \\ k_1(0.6)^t \end{bmatrix}$$

For $\qquad r_2 = 0.1$,

$$\begin{bmatrix} 0.4-0.1 & 0.6 \\ 0.1 & 0.3-0.1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.6 \\ 0.1 & 0.2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$0.3c_1 + 0.6c_2 = 0 \qquad c_1 = -2c_2$

If $\quad c_2 = 1$, $c_2 = -2$, and

$$k_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}(0.1)^t = \begin{bmatrix} -2k_2(0.1)^t \\ k_2(0.1)^t \end{bmatrix}$$

Combining the two for the general complementary functions,

$$x_c = 3k_1(0.6)^t - 2k_2(0.1)^t$$

$$y_t = k_t(0.6)^t + k_2(0.1)^t$$

4. For the particular solution,

$$y_p = (\mathrm{I}-\mathrm{A})^{-1}\,\mathrm{B}$$

where $(\mathrm{I}-\mathrm{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.6 \\ 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.6 \\ -0.1 & 0.7 \end{bmatrix}$

and $y_p = \begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix} = \dfrac{1}{0.36}\begin{bmatrix} 0.7 & 0.6 \\ 0.1 & 0.6 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

This makes the complete general solution

$$x_t = 3k_t(0.6)^t - 2k_2(0.1)^t + 20$$

$$x_t = k_1(0.6)^t + k_2(0.1)^t + 10 \qquad (19.33)$$

5. Imploying the initial conditions, $x_0 = 14, y_0 = 23$, (19.33) reduces to

$$3k_1 - 2k_2 + 20 = 14$$

$$k_1 + k_2 + 10 = 23$$

Solved simultaneously, $k_1 = 4$, $k_2 = 9$

Substituting in (19.33),

$$x_t = 12(0.6)^t - 18\mathrm{k}_2(0.1)^t + 20$$

$$y_t = 4(0.6)^t + 9\ (0.1)^t + 10$$

With $|0.6| < 1$ and $|0.1 < 1$, the time path is convergent. With both roots positive, there will be no oscillation.

Solve the following system of first-order linear difference equations.

$$x_t = -\ 0.6x_{t-1} + 0.y_{t-1} + 9 \qquad x_0 = 7.02$$

$$y_\mathrm{t} = 0.5_{t-1} - 0.2\ y_{t-1} + 42 \qquad y_0 = 57.34$$

1. In matrix form,

$$\begin{bmatrix} x_t \\ x_t \end{bmatrix} = \begin{bmatrix} -0.6 & 0.1 \\ 0.5 & -0.2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} 9 \\ 42 \end{bmatrix}$$

$$Y_t = AY_{t-1} + B$$

2. The characteristic roots are

$$r_1, r_2 = \frac{-0.8 \pm \sqrt{(-0.8)^2 - 4(0.07)|}}{2} = \frac{-0.8 \pm 0.6}{2}$$

$$r_1 = -0.1 \qquad r_2 = -0.7$$

3. The eigenvector for $r_1 = -0.1$ is

$$\begin{bmatrix} -0.6-(0.1) & 0.1 \\ 0.5 & -0.2-(0.1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.1 \\ 0.5 & -0.1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-0.5c_1 + 0.1c_2 = 0 \qquad c_2 = 5c_1$$

If $c_2 = 1, c_2 = 5,$ and

$$k_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} (-0.1)^t = \begin{bmatrix} k_1(-0.1)^t \\ 5k_1(-0.1)^t \end{bmatrix}$$

For $r_2 = -0.7,$

$$\begin{bmatrix} -0.6-(0.7) & 0.1 \\ 0.5 & -0.2-(0.7) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$0.1c_1 + 0.1c_2 = 0 \qquad c_1 = -c_2$$

If $c_2 = 1,\ c_2 = -1$ and

$$k_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (-0.7)^t = \begin{bmatrix} -k_2(-0.7)^t \\ k_2(0-7)^t \end{bmatrix}$$

This makes the general complementary functions,

$$x_c = k_1(0.1)^t - k_2(-0.7)^t$$

$$y_c = 5k_1(-0.1)^t + k_2(-0.7)^t$$

4. For the particular solution,

$$y_p = (\mathrm{I} - \mathrm{A})^{-1}\,\mathrm{B}$$

where $(\mathrm{I} - \mathrm{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -0.6 & 0.1 \\ 0.5 & -0.2 \end{bmatrix} = \begin{bmatrix} 1.6 & -0.1 \\ 0.5 & 1.2 \end{bmatrix}$

and $\quad y_p = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \dfrac{1}{1.87}\begin{bmatrix} 1.2 & 0.1 \\ 0.5 & 1.6 \end{bmatrix}\begin{bmatrix} 9 \\ 42 \end{bmatrix} = \begin{bmatrix} 8.02 \\ 38.34 \end{bmatrix}$

This makes the complete general solution,

$$x_1 = k_1(-0.1)^t - k_2(-0.7)^t + 8.02$$

$$y_2 = 5k_1(-0.1)^t + k_2(-0.7)^t + 38.34 \quad (19.34)$$

5. Using 7.02, $y_0 = 57.34$, (19.34) reduces to

$$k_1 - k_2 + 8.02 = 7.02$$

$$5k_1 + k_2 + 38.34 = 57.34$$

Solved simultaneously, $\quad k_1 = 3 \quad\quad k_2 = 4$

Substituting in (19.34)

$$x_t = 3(-0.1)^t - 4(-0.7)^t + 8.02$$

$$y_t = 15(-0.1)^t + 4(-0.7)^t + 38.34$$

With both characteristic roots in absolute value less than 1, the system of equations will approach a stable temporal equilibrium solution. With the roots negative, there will be oscillation.

Solve the following system of first-order linear difference equations in which one difference is a function of another difference.

$$x_t = 0.7x_{t-1} - 0.4y_{t-1} + 40 \qquad x_0 = 24$$

$$y_t = 0.575x_{t-1} - 0.5y_{t-1} - x_t + 6 \qquad y_0 = -32$$

1. Rearranging and setting in matrix form,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_t \\ x_t \end{bmatrix} = \begin{bmatrix} -0.7 & -0.4 \\ -0.575 & -0.5 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} 40 \\ 6 \end{bmatrix}$$

$$A_1 Y_t = A_2 Y_{t-1} + B$$

2. We then find the characteristic roots from the characteristic equation,

$$|A_2 - r_i A_1| = 0$$

$$\begin{bmatrix} -0.7 & -0.4 \\ -0.575 & -0.4 \end{bmatrix} - r_1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -0.7 - r_1 & -0.4 \\ -0.575 - r_1 & -0.5 - r_1 \end{bmatrix} = 0$$

$$r_1 = -0.6 \qquad r_2 = -0.2$$

3. The eigenvector for $r_1 = -0.6$ is

$$\begin{bmatrix} -0.7 - (0.6) & -0.4 \\ -0.575 - (-0.6) & -0.5 - (0.6) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.025 & 0.1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-0.1c_1 - 0.4c_2 = 0 \qquad c_1 = -4c_2$$

If 1, $c_1 = -4$ and the eigenvector is

$$k_1\begin{bmatrix} -4 \\ 1 \end{bmatrix}(-0.6)^t = \begin{bmatrix} -4k_1(-0.6)^t \\ k_1(-0.6)^t \end{bmatrix}$$

For $r_2 = -0.2$,

$$\begin{bmatrix} -0.7 - (0.2) & -0.4 \\ -0.575 - (-0.2) & -0.5 - (0.2) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.5 & -0.4 \\ -0.375 & 0.3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-0.5c_1 - 0.4c_2 = 0 \qquad\qquad c_2 = -1.25c_1$$

$c_1 = 1$, $c_2 = -1.25$, and the eigenvector for $r_2 = -0.2$ is

$$k_2 \begin{bmatrix} 1 \\ -1.25 \end{bmatrix} (-0.2)^t = \begin{bmatrix} k_2(-0.2)^t \\ -1.25k_2(-0.2)^t \end{bmatrix}$$

Adding the two eigenvectors, the complementary functions are

$$x_c = -4k_1(-0.6)^t + k_2(-0.2)^t$$

$$y_c = k_1(-0.6)^t - 1.25k_2(-0.2)^t$$

4. For the particular solution $y_p = \vec{\text{Y}}$,

$$\vec{\text{Y}} = (\text{A}_1 - \text{A}_2)^{-1}\text{B}$$

$$\text{where } (\text{A}_1 - \text{A}_2) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} -0.7 & -0.4 \\ -0.575 & -0.5 \end{bmatrix} = \begin{bmatrix} 1.7 & 0.4 \\ 1.575 & 1.5 \end{bmatrix}$$

and

$$\vec{\text{Y}} = \begin{bmatrix} \vec{x} \\ \vec{x} \end{bmatrix} = \frac{1}{1.92} \begin{bmatrix} 1.5 & -0.4 \\ -1.575 & 1.7 \end{bmatrix} \begin{bmatrix} 40 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ -27.5 \end{bmatrix}$$

Addint y_c and y_p the complete general solution is

$$x_1 = -4k_1(-0.6)^t + k_2(-0.2)^t + 30$$

$$y_t = k_1(-0.6)^t - 1.25k_2(-0.2)^t - 27.5 \qquad 19.35$$

5. Finally, we apply the initial conitions $x_0 = 24$ and $y_0 = -32$, to (19.35)

$$-4k_1 + k_2 + 30 = 24$$

$$k_1 - 1.25k_2 - 27.5 = -32$$

$$k_1 = 3 \qquad k_2 = 6$$

and substitute these values back in (19.35) for the definite solution

$$x_t = -12\,(-0.6)^t + 6\,(-0.2)^t + 30$$

$$y_t = 3(-0.6)^t - 7.5\,(-0.2)^t - 27.5$$

With both characteristic roots less than 1 in absolute value, the solution is stable.

Solve the following system of first-order linear difference equations.

$$x_t = 0.6x_{t-1} + 0.85y_{t-1} - y_t + 15 \qquad x_0 = 27$$

$$y_t = 0.2x_{t-1} + 0.4y_{t-1} + 6 \qquad y_0 = 38$$

1. In matrix form,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.85 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

$$A_1 Y_t = A_2 Y_{t-1} + B$$

2. For the characteristic roots, $|A_2 - r_i A_1| = 0$

$$\begin{bmatrix} 0.6 & 0.85 \\ 0.2 & 0.4 \end{bmatrix} - r_i \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 - r_i & 0.85 - r_i \\ 0.2 & 0.4 - r_i \end{bmatrix} = 0$$

$$r^2 - 0.8\,r + 0.07 = 0$$

$$r_1 = 0.7 \qquad r_2 = 0.1$$

3. For $r_1 = 0.7$

$$\begin{bmatrix} 0.6 - 0.7 & 0.85 - 0.7 \\ 0.2 & 0.4 - 0.7 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.15 \\ 0.2 & -0.3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-0.1c_1 + 0.15c_2 = 0 \qquad c_1 = 1.5c_2$$

If $c_2 = 1 = 1.5$ and the eigenvector is

$$k_1\begin{bmatrix}1.5\\1\end{bmatrix}(0.7)^t=\begin{bmatrix}1.5k_1(0.7)^t\\k_1(0.7)^t\end{bmatrix}$$

For $r_2 = 0.1$. $\begin{bmatrix}0.6-0.1 & 0.85-0.1\\0.2 & 0.4-0.1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0.5 & 0.75\\0.2 & 0.3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=0$

$0.5c_1+0.75c_2=0$ $\qquad\qquad$ $c_1=-1.5c_2$

If $c_2 = 1 = -1.5$ and the eigenvector is

$$k_2\begin{bmatrix}-1.5\\1\end{bmatrix}(0.1)^t=\begin{bmatrix}-1.5k_2(0.1)^t\\k_2(0.1)^t\end{bmatrix}$$

4. For the particular solution,

$$\vec{Y}=(A_1-A_2)^{-1}B$$

Here $(A_1-A_2)=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}-\begin{bmatrix}0.6 & 0.85\\0.2 & 0.4\end{bmatrix}=\begin{bmatrix}0.4 & 0.15\\-0.2 & 0.6\end{bmatrix}$

and $\quad \vec{Y}=\begin{bmatrix}\vec{x}\\\vec{x}\end{bmatrix}\frac{1}{0.27}\begin{bmatrix}0.6 & -0.15\\0.2 & 0.4\end{bmatrix}\begin{bmatrix}15\\6\end{bmatrix}=\begin{bmatrix}30\\20\end{bmatrix}$

Adding y_t and y_p,

$$x_t = 1.5k_t(0.7)^t-1.5k_2(0.1)^t+30 \qquad (19.36)$$

$$y_t = k_1(0.7)^t+k_2(0.1)^t+20$$

5. For the definite solution, we apply $x_0 = 27$ and $y_0 = 38$ to (19.36),

$1.5\ k_t - 1.5\ k_2 + 30 = 27$

$k_1 + k_2 + 20 = 38$

$k_1 = 8$ $\qquad\qquad$ $k_2 = 10$

Substituting back in (19. 36)

$$x_t = 12(0.7)^t - 15(0.1)^t + 30$$

$$k_t = 8(0.7)^t + 10(0.1)^t + 20$$

With both characteristic roots less than 1 in absolute value, the solution is stable.

13.5 PHASE DIAGRAMS FOR SIMULTANEOUS DIFFERENTIAL EQUATIONS

Example : Given the system of linear autonomous differential equations,

$$y_1 = -4y_1 + 16$$

$$y_1 = -5y_2 + 15$$

A phase diagram is used below to test the stability of the model. Since neither variable is a function of the other variable in this simple model, each equation can be graphed separately.

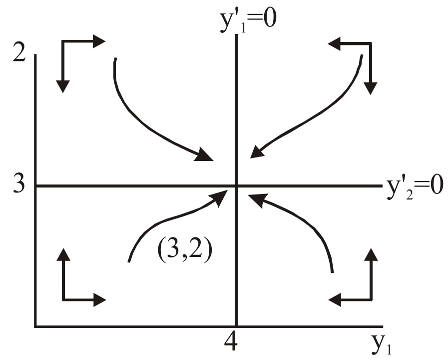
1. Find the intertemporal equilibrium level, $\vec{y}_1$ i.e., the locus of points at which $y_1 = 0$.

$$y_1 = -4y_1 + 16 = 0 \quad \vec{y}_1 = 4$$

The graph of $\vec{y}_1 = 4$, a vertical line at $y_1 = 4$, is called the y_1 isocline. The y_1 isocline divides the phase plane into two regions called isosectors, one to the left of the y_1 isoscline and one to the right.

2. Find the intertemporal equilibrium level, $\vec{y}_2$, i.e. the locus of points at which $y_2 = 0$.

$$y_2 = -5y_2 + 15 = 0 \qquad \vec{y}_2 = 3$$



The graph of $\vec{y}_2 = 3$ is a horizontal line at $y_2 = 3$, called the y_2 isocline. The y_2 isocline divides the phase plane into two isosectors, one above the y_2 isocline and the other below it. See Fig. 19.1

The intersection of the isoclines demarcates the intertemporal equilibrium level.

$$(\vec{y}_1, \vec{y}_2) = (4,3)$$

3. Determine the motion around the y_1 isocline, using arrows of horizontal motion.

   (a) To the left of the y_1 isocline, $y_1 < 4$.

   (b) To the right of the y_1 isocline, $y_1 > 4$.

   By substituting these values successively in $y_1 = -4y_1 + 16$, we see

   If $y_1 < 4$, $y_1 > 0$, and there will be motion to right

   If $y_1 > 4$, $y_1 < 0$, and there will be motion to left.

4. Determine the motion around the y_2 isocline, using arrows of vertical motion.

   (a) Above the y_2 isocline, $y_2 > 3$.

   (b) Below the y_2 isocline, $y_2 < 3$.

   Substitution of these values successively in $y_2 = -5y_2 + 15$, shows

If $y_2 > 3$, $y_2 < 0$, and the motion will be downward.

If $y_2 < 3$, $y_2 > 0$, and the motion will be upward.

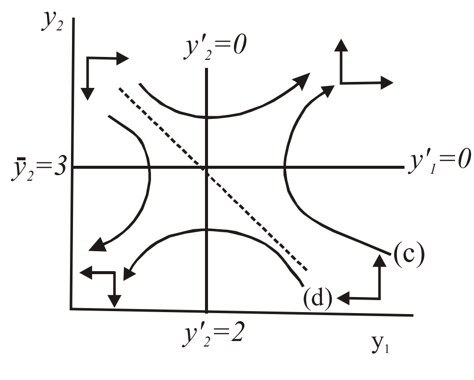
The resulting arrows of motion in Fig. 19.1, all pointing to the intertemporal equilibrium, suggest the system of equations is convergent. Nevertheless, trajectory paths should be drawn because the arrows by themselves can be deceiving, as seen in Fig. 19.2. Starting from an arbitrary point, such as (3, 2) in the southwest quadrant, or any point in any quadrant, we can see that the dynamics of the model will lead to the steady-state solution (4, 3). Hence the time path converges to the steady-state solution, making that solution stable.

Since the equations are linear, the answer can be checked using the techniques of Chapter 16 or 19, getting

$$y_1(t) = k_1 e^{-4t} + 4$$

$$y_2(t) = k_2 e^{-5t} + 3$$

With both characteristic roots negative, the system must be stable.



Example : A phase diagram is constructed in Fig. 19.2 and used below to test the dynamic stability of a saddle point equilibrium for the system of equations.

$$y_1 = 2y_2 - 6$$

$$y_2 = 8y_1 - 16$$

1. Find the y_1 isocline on which $y_1 = 0$.

$y_1 = 2y_2 - 6 = 0 \qquad \vec{y}_2 = 3$

Here the y_1 isocline is a horizontal line at $\vec{y}_2 = 3$.

2. Find the y_2 isocline on which $y_2 = 0$.

   $y_2 = 8y_1 - 16 \qquad \vec{y}_1 = 2$.

   The y_2 isocline is a vertical line at $\vec{y}_1 = 2$. See Fig. 19.2.

3. Determine the motion around the y_1 isocline, using arrows of horizontal motion.

   (a) Above the y_1 isocline $y_2 > 3$. (b) Below the y_1 isocline, $y_2 < 3$.

   Substitution of these values successively in $y_1 = 2y_2 - 6$, shows

   If $y_2 > 3$, $y_1 > 0$, and the arrows of motion point to the right.

   If $y_2 < 3$, $y_1 < 0$, and the arrows of motion point to the left.

4. Determine the motion around the y_2 isocline, using arrows of vertical motion.

   (a) To the left of the y_2 isocline, $y_1 < 2$.

   (b) To the right of the y_2 isocline, $y_1 > 2$.

   By substituting these values successively in $y_2 = 8y_1 - 16$, we see

   If $y_1 < 2$, $y_2 < 0$, and there will be motion downward.

   If $y_1 > 2$, $y_2 > 0$, and there will be motion upward.

Despite appearances in Fig. 19.2, the system is unstable even in the northwest and southeast quadrants. As explained in Example 11, we can show by simply drawing trajectories that the time paths diverge in all four quadrants, whether we start at point a, b, c, or d.

Example : The instability in the model in Fig. 19.2 is made evident by drawing a trajectory from any of the quadrants. We do two, one from a and one from b, and leave the other two for you as a practice exercise. In each case the path of the trajectory is best described in four steps.

1. Departure from piont a.

   (a) The trajectory moves in a southeasterly direction.

   (b) But as the time path approaches the y_1 isocline where $y_1 = 0$, the y_1 motion eastward slows down while the y_2 motion southward continues unabated.

   (c) At the y_1 isocline, $y_1 = 0$. Consequently, the trajectory must cross the y_1 isocline vertically.

   (d) Below the y_1 isocline, the arrows of motion point in a southwesterly direction, taking the time path away from the equilibrium and hence indicating an unstable equilibrium.

2. Departure from point *b*.

   (a) The trajectory once again moves in a southeasterly direction.

   (b) But as the time path approaches the y_2 isocline where $y_2 = 0$, the y_2 motion southward ebbs while the y_1 motion eastward continues unaffected.

   (c) Since $y_2 = 0$ at the y_2 isocline, the time path must cross the y_2 isocline horizontally.

   (d) To the right of the y_2 isocline, the arrows of motion point in a northeasterly direction, taking the time path away from the equilibrium and belying the appearance of a stable equilibrium.

Example : The dotted line in Fig. 19.2 is a saddle path. Only if the initial conditions fall on the saddle path will the steady-state equilibrium prove to be stable. The equation for the saddle path is

$$y_2 = \left(\frac{r_1 - a_{11}}{a_{12}}\right)(y_1 - \bar{y}_1) + \bar{y}_2$$

Where we already know all but r_1, the negative root. From the original equations.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 8 & 0 \end{vmatrix} = -16$$

Any time $|A| < 0$, we have a saddle-point equilibrium. Substituting in (19.3),

$$r_1 = \frac{\text{Tr (A)} \pm \sqrt{[\text{Tr(A)}]^2 - 4|A|}}{2}$$

$$|A_2 - r_1 A_1| = 0$$

Then substituting in the saddle-path equation above,

$$y_2 = \left(\frac{4-0}{2}\right)(y_1 - 2) + 3$$

$$y_2 = 7 - 2y_1 \quad \text{saddle path}$$

Note that the intertemporal equilibrium (2, 3) falls on the saddle path. Only if the initial conditions satisfy the saddle-path condition will the intertemporal equilibrium be stable.

Use a phase diagram to test the stability of the system of equations,

$$y_1 = 3y_1 - 18$$

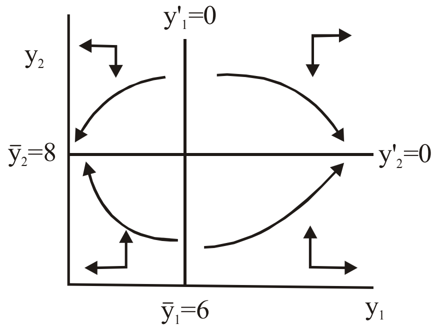
$$y_2 = -2y_2 + 16$$

1. Determine the steady-state solutions y_t where $y_t = 0$ to find the isoclines.

$$y_1 = 3y_1 - 18$$

$$y_2 = -2y_2 + 16$$

$\vec{y}_1 = 6$ the y_1 isocline $\qquad$ $\vec{y}_2 = 8$ the y_2 isocline



As soon in Fig. 19.3, the intersection of the isoclines demarcates the intertemporal equilibrium level, $(\bar{y}_1, \bar{y}_2) = (6, 8)$.

2. Determine the motion around the y_1 isocline using arrows of horizontal motion.

   (a) To the left of the y_1 isocline, $y_1 < 6$.

   (b) To the right of the y_1 isocline, $y_1 > 6$.

   Substituting these values successively in $y_1 = 3y_1 - 18$, we see

   If $y_1 < 6$, $y_1 < 0$ and there will be motion to the left.

   If $y_1 < 6$, $y_1 > 0$, and there will be motion to the right.

3. Determine the motion around the y_2 isocline, using arrows of vertical motion.

   (a) Above the y_2 isocline, $y_2 > 8$. (b) Below the y_2 isocline, $y_2 < 8$.

   Substitution of these values successively in $y_2 = -2y_z + 16$ shows

   If $y_2 > 8$, $y_2 < 0$, and the motion will be downward.

   If $y_2 < 8$, $y_2 > 0$, and the motion will be upward.

The resulting arrows of motion in Fig. 19.3 all pointing away from the intertemporal equilibrium, suggest the system of equations is divergent. Drawing trajectory paths to be sure confirms that the system is indeed divergent.

Proots and Demonstrations

Given $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

or $\dot{Y} = AY + B$ (19.37)

Show in terms of Section 19.1 and Example 1 that

$$(A - r_1 I)\ C_i = 0$$

Starting with the homogeneous form of the system of equations in which B = 0, or a null vector, and assuming distinct real roots, we can expect the solution to be in the form.

$$Y = k_i C_i e^{r_i t} \qquad (19.38)$$

where k_i = a scalar, C_i = (2 × 1) column vector of constants, and r_i = a scalar. Taking the dervative of (19.38) with respect to *t,* we have

$$\dot{Y} = r_i k C_i e^{r_i t} \qquad (19.39)$$

Substituting (19.38) and (19.39) in the homogeneous form of (19.37) where B=0.

$$r_i k_i C_i e^{r_i t} = A k_i C_i e^{r_i t}$$

Canceling the common k_i and $e^{r_i t}$ terms, we have

$$r_1 C_i = AC$$

$$AC_i = r_1 C_i = 0$$

Factoring out C_i and recalling that A is a (2 × 2) matrix while r_i is a scalar, we multiply r_i by a (2×2) identify matrix I_2, or simply I, to

$$(A - r_i I)\ C_i = 0 \qquad \text{Q.E.D.} \qquad (19.40)$$

Continuing with the model in Problem 19.12,

show that $$r_1 r_2 = \frac{Tr(A) \pm \sqrt{[Tr(A)]^2 - 4|A|}}{2}$$

If $(A - r_i I)$ is nonsingular in (19.40), meaning it contains no linear dependence, then C_i must be a null column sector, making the solution trivial. To find a nontrivial solution, $(A - r_i I)$ must be singular. A necessary condition for a nontrivial solution $(C_r \neq 0)$, then, is that the determinant.

$$| A - r_1 I | = 0 \tag{19.41}$$

where equation (19.41) is called the characteristic equation or characteristic polynomial for matrix A. Dropping the subscript for simplicity and substituting from above, we have

$$\begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0$$

$$a_{11}a_{22} - a_{11}r - a_{22}r + r^2 - a_{12}a_{21} = 0$$

Rearranging $\quad r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0$

Or, using matrix motion,

$$r^2 - \mathrm{T}r\,(A)\, r + |A| = 0$$

which is a quadratic equation that can be solved for r with the quadratic formula.

$$r_1 r_2 = \frac{\mathrm{T}r(A) \pm \sqrt{[\mathrm{T}r(A)]^2 - 4|A|}}{2} \quad \text{Q.E.D.}$$

Continuing with the model in Problem 19.13, show that the particular integral or solution is

$$y_p = \vec{Y} = -A^{-1}B \tag{19.42}$$

The particular integral is simply the intertemporal or steady-state solution $\vec{Y}$. To find the steady-state solution, we simply set the column vector of derivatives equal to zero such that $\dot{Y} = 0$. When $\dot{Y} = 0$, there is now change $Y = \vec{Y}$. Substituting in (19.37),

$$\dot{Y} = A\vec{Y} + B = 0$$

$$A\vec{Y} = -B$$

$$\vec{Y} = -A^{-1}B \quad \text{Q.E.D.}$$

Given $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Or $$A_1\dot{Y} = A_2Y + B \qquad (19.43)$$

Draw in terms of Section 19.2 and Example 3 that to find the complementary function, one must solve the specific eigenvalue problem

$$(A_2 - r_1A_1)\ C_1 = 0$$

Starting with the homogeneous form of (19.43) in which B = 0, or a null vector, and assuming distinct real roots, we can expect the solution and its derivative to take the forms

$$Y = k_iC_ie^{r_it} \qquad \dot{Y} = r_ik_iC_ie^{r_it} \qquad (19.44)$$

Substituting from (19.44) into the homogeneous form of (19.43) where B = 0,

$$A_1r_ik_iC_ie^{r_it} = A_2k_iC_ie^{r_it}$$

Canceling the common k_i and e^{r_it} terms, we have

$$A_1r_iC_i = A_2C_i$$

$$(A_2 - r_iA_1)C_i = 0 \quad \text{Q.E.D.}$$

In terms of the model in Problem 19.15, show that the particular integral is

$$y_p = \bar{Y} = -A_2^{-1}B \qquad (19.45)$$

The particular inegral is the steady-state solution $\bar{Y}$ when $\dot{Y} = 0$. Substituting in (19.43).

$$A_1\dot{Y} = A_2\bar{Y} + B = 0$$

$$A_2\bar{Y} = -B$$

$$\bar{Y} = -A_2^{-1}B \qquad \text{Q.E.D}$$

Given
$$\begin{bmatrix} x_t \\ x_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or
$$Y_t = AY_{t-1} + B \qquad (19.46)$$

Show in terms of Section 19.3 and Example 5 that the eigenvalue problem for a system of simultaneous first-order linear difference equations when no difference is a function of another difference is

Starting with the homogeneous form of the system of equations in which B = 0, and assuming a case of distinct real roots, from what we know of individual difference equations, we can expect that

$$Y_t = k_i C_i (r_i)^t \qquad \text{and} \qquad Y_{t-1} = k_i C_i (r_i)^{t-1} \qquad (19.47)$$

which k_i and r_i are scalars, and C_i = (2 × 1) column vector of constants. Substituting in (19.46) when B = 0, we have

$$k_i C_i (r_i)^t = A k_i C_i (r_i)^{t-1}$$

Canceling the common k_i terms and rearranging,

$$AC_i (r_i)^{t-1} - C_i (r_i)^t = 0$$

Evaluated at $t = 1$.

$$(A - r_i I)\, C_i = 0 \quad \text{Q.E.D.}$$

Given
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_i \\ x_i \end{bmatrix} = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} x_i - 1 \\ x_i - 1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A_1 Y_t = A_2 Y_{t-1} + B \qquad (19.48)$$

show in terms of Section 19.4 and Example 9 that for a system of simultaneous first-order linear difference equations when one or more difference is a function of another difference the eigenvalue problem is

$$(A_2 - r_i A_1)C_i = 0$$

From earlier work, assuming distinct real roots, we can anticipate

$$Y_t = k_i C_i (r_i)^t \text{ and } Y_{t-1} = k_i C_i (r_i)^{t-1} \qquad (19.49)$$

Substituting in the homogeneous form of (19.48) where B = 0, we have

$$A_1 k_i C_i (r_i)^t = A_2 k_i C_i (r_i)^{t-1}$$

$$A_2 C_i (r_i)^{t-1} - A_1 C_i (r_i)^t = 0$$

Evaluated at 1,

$$(A_2 - r_i A_i)\ C_i = 0 \qquad \text{Q.E.D}$$

Remaining with the same model as in Problem 19.18, show that the particular solution is

$$y_p = \vec{Y} = (A_1 - A_2)^{-1}\ B \qquad (19.50)$$

For the particular or steady-state solution.

$$x_t = x_{t-i} = \vec{x} \text{ and } \qquad y_t = y_{t-1} = \vec{y}$$

In matrix notation, $Y_t = Y_{t-1} = \vec{Y}$

Substituting in (19.48)

$$A_1 \vec{Y} = A_2 \vec{Y} + B$$

Solving for $\vec{Y}$

$$\vec{Y} = (A_1 - A_2)^{-1}\ B \qquad \text{Q.E.D.}$$

13.6 SUMMARY

We end this lesson by summarizing what we have covered in it.

i) Simultaneous Differential Equations.

ii) Simultaneous Difference Equations.

iii) Phase Diagrams for simultaneous Differential Equations.

13.7 LESSON END EXERCISE

1. $\dot{y}_1 = 5y_1 - 0.5y_2 - 12$ $\quad y_1(0) = 12$

   $\dot{y}_2 = -2y_1 + 5y_2 - 24$ $\quad y_2(0) = 4$

2. $\dot{y}_1 = 3y_1 - 1.5y_2 - 2.5\dot{y}_2 + 24$ $\quad y_1(0) = 14$

   $\dot{y}_2 = 2y_1 - 5y_2 + 16$ $\quad y_2(0) = 15.4$

3. $x_t = -4x_{t-1} + y_{t-1} + 12$ $\quad x_0 = 16$

   $y_t = 2x_{t-1} + 3y_{t-1} + 6$ $\quad y_0 = 16$

13.8 SUGGESTED READINGS

Black. J. and Bradley. J. F. Essential Mathematics for Economists.

Henderson J. M. and Quandt. R. E. Microeconomic Theory : A Mathematical Approach.

Aggarwal, C.S & R.C Joshi : Mathematics for students of Economics (New Academic Publishing Co.)

Kandoi B. : Mathematics and Economic Will Applications (Himalaya Publishing House).

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APPLICATIONS OF DIFFERENCE AND DIFFERENTIAL EQUATIONS IN ECONOMICS

Structure

14.1 INTRODUCTION

Applications of difference and differential equations are now used in mideling motion and change in all areas of science. The theory of these equations has become an essential tool of economic analysis particularly since computers have become commonly avilable. It would be difficult to comprehend to coctemporary literature of economics if one doesn't understand the basic concepts and the results of modern theories of difference and differential equations.

14.2 OBJECTIVES

After reading this unit you should be able to apply the difference and differential equations in Economics by understanding the following economic models :

i) Cocweb Model

ii) Foreign Trade Multiplier Model

iii) Capital Stock Adjustment Theory of Investment

iv) Market Model with stocks in case of Difference and differential Equations

v) National Incomes Model.

14.3 COBWEB MODEL OR LAG ADJUSTED MODEL

Consider a situation in which the producer's output decision must be made one period in advance of the actual sale – such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output. Let us assume that the output decision in period t is based on the then prevailing price P_t. Since this output will not be available for the sale until period $(t + 1)$, however, P_t will determine not Q_{st}, but $Q_{s,t+1}$. Thus, we now have a "lagged" supply function.

$$Q_{s,t+1} = S(P_t).$$

or, equivalently, by shifting back the time subscripts by one period,

$$Q_{st} = S(P_{t-1})$$

when such a supply functions interacts with a demand function of the form,

$$Q_{dt} = D(P_t)$$

interesting dynamic price patterns will result,

Taking the linear versions of these lagged supply and unlagged demand functions, and assuming that in each time the market price is always set at a level which clears the market, we have a market model with the following three equations.

$$Q_{dt} = Q_{st} \qquad \text{...(1)}$$

$$Q_{dt} = \square - \square pt \ (\square, \square > 0) \qquad \text{...(2)}$$

$$Q_{st} = -\,\square + \square p_{t-1} \;(\square, \square > 0) \quad ...(3)$$

By substituting the last two equations into first, the model can be reduced to a single first order difference equation as follows :

$$\beta p_t + \delta P_{t-1} = \alpha + \gamma$$

$$(i.e.) \quad P_t + \frac{d}{b} P_{t-1} \quad \frac{d+g}{b}$$

□ The particular solution :

$$\bar{P} + \frac{d}{b}\bar{p} = \frac{a+g}{b}$$

$$\square \quad \bar{P}\frac{(b+d)}{b} = \frac{a+g}{b}$$

$$\square \quad \bar{P} = \frac{(a+g)}{b} , \frac{b}{b+d}$$

$$\square \quad yp = \frac{a+g}{b+g}$$

□ The complementary solution

$$\text{Put,} \quad p_t = Ar^t, \quad P_{t-1} = \frac{Ar^t}{r}$$

$$\square \quad Ar^t + \frac{d}{b}\frac{Ar^t}{r} = 0$$

$$(i.e) \quad r + \frac{d}{b} = 0$$

$$(i.e.) \quad r = -\frac{d}{b}$$

□ $y_c = -\frac{d}{b}$

□ General solution

$$P_t = \overline{P} + A\left(-\frac{d}{b}\right)^t$$

(i.e.) $P_t = \frac{a + g}{b + d} + A\left(-\frac{d}{b}\right)^t$

□ the definite solution

(i.e.) $Pt = \frac{a + 8}{b + 8} + \left(P_0 - \overline{P}\right)\left(-d / b\right)^t$

where, P_0 represents the initial price

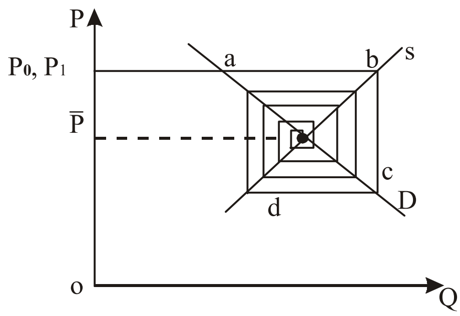
where, □ slope of supply curve,

□ slope of demand curve,

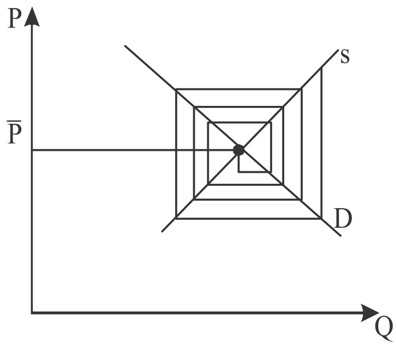
□ There arises three possible varieties of oscilation patterns in the model, (i.e.)

δ < β

Explosive
Uniform if
Damped



Or, Here, if



$\frac{d}{b} > 1$ $\substack{9\,23\\8\,11}$ divergence

(*i.e.*) □> □

$\frac{d}{b} < 1$ $\substack{9\,23\\8\,11}$ convergence

(*i.e.*) □ < □

Q. Solve $(x + 2y^3)\, dy = ydx$

Sol. $(x + 2y^3)dy = ydx$

$$□\ (x + 2y^3)\frac{dy}{dx} = y$$

$$□\ \frac{x + 2y^2}{y} = \frac{dx}{dy}$$

$$□\ \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

Solve next,

$$x = y^3 + cy$$

Q. $\frac{dy}{dx} = \frac{1+y}{1+x}$

Hint. $\frac{dy}{dx} - \frac{y}{1+x} = \frac{1}{1+x}$

Sol. $y = -1 + c\,(x + 1)$

Q. Solve, $\frac{dy}{dx} + \frac{1-2x}{x^2}\,y = 1$

Ans. $y = x^2\ (1 + ce^{1/x})$

Q. Solve $(x^2 + 1)\ \dfrac{dy}{dx} + 2xy = 4x^2$

Sol. $3(x^2 + 1)\ y = 4x^3 + c$

Q. $\dfrac{dy}{dx} + y = 3t$

Sol. Given equation is

$$\frac{dy}{dt} + y = 3t$$

Here, $a = \wedge$, $b = 3t$

$$\int a.dt = \int 1.dt = t$$

I.F $= e^{sp.dt} = e^t$

Multiplying b/s by e^t, we get

$$e^t\left[\frac{dy}{dt} + y\right] = e^t.3t$$

$$\frac{d}{dt}[e^t.y] = e^t.3t$$

Integrating , we get

$$\begin{aligned} e^t.y &= \int e^t.3t.dt \\ &= 3\int t.e^t dt \\ &= 3\ t.e^t - 3e^t + c \\ e^t.y &= 3e^t\ (t - 1) + c \\ \therefore\ y &= (t-1) + ce^{-t} \end{aligned}$$

Q. $\frac{dy}{dx}+\frac{3}{x}y=\frac{a}{x^3}$

Sol. $yx^3 = ax + c$

Q. $\frac{dy}{dx}+y=e^{-x}$

Sol. $ye^x = x + c$

Q. $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$

Sol. $\frac{dy}{dx}+\frac{2x}{x^2+1}.y = \frac{4x^2}{x^2+1}$

Here, $P = \frac{2x}{x^2+1}$, $Q = \frac{4x^2}{x^2+1}$

$$\int P.dx = \int\frac{2x}{x^2+1}.dx = \log (x^2 + 1)$$

$\therefore$ integration factor, $e^{\int Pdx} = e^{\log(x^2+1)} = x^2+1$

$\therefore$ Solution is

$$y(x^2 + 1) = \int\frac{4x^2}{(x^2+1)}.(x^2+1)dx+c$$

$$= \int 4x^2 dx + c$$

$$= \frac{4}{3}x^3 + c$$

$\therefore$ $3(x^2 + 1)\, y = 4x^3 + c$

DIFFERENCE EQUATIONS IN TWO VARIABLES

The technique of second-order difference equations can be used to analyse models in which two variables are each single lagged functions of themselves and of each other.

14.4 FOREIGN TRADE MULTIPLIER MODEL

Suppose we take two countries and assume that Y is income in country 1 and R is income in country 2. Taking country 1 we have

$$Y_t = C_t + I_t + G_t + X_t - M_t$$

Assume that

$$C_t = C^* + cY_{t-1}$$

$$I_t = I^*$$

$$G_t = G^*$$

and

$$M_t = M^* + .00Y_{t-1}$$

where $.00 > 0$ is the propensity to import and C^*, I^*, G^* and M^* are the autonomous parts of the relevant functions.

If country 1's exports X_t are a function of country 2's income lagged one period then

$$X_t = aR_{t-1}$$

where $a > 0$. This gives

$$Y_t = C^* + I^* + G^* - M^* + (c - .00)Y_{t-1} + aR_{t-1}$$

if we let

$$k = C^* + I^* + G^* - M^*$$

and $$b = c - .00$$

then $$Y_t = k + b\,Y_{t-1} + aR_{t-1} \qquad (1)$$

This is a first-order linear difference equation in two variables, Y and R, where Y is determined by a single lag as a function of R and Y.

Taking country 2 we have

$$R = C'_t + I'_t + G'_t + X'_t - M'_t$$

Since country 2's exports are the same as country 1's imports we have

$$X'_t = M_t = M^* + mY_{t-1}$$

and the other functions are as above, then the identity can be reduced to the form

$$R_t = l + gR_{t-1} + m + Y_{t-1} \qquad (2)$$

where l stands for the autonomous components of the functions, see k above, and $g = c' - a$, similar to b above.

We now have two first-order difference equations, *i.e.* equations (1) and (2), in two variables where each is a single lagged function of itself and of the other. It is possible to examine the time paths of Y and R using the same methods as those employed to deal with difference equations in a single variable.

Equilibrium Income Levels

If there is an equilibrium income level in each country, these levels are such that if both are attained simultaneously they will both be repeated, i.e. if

$$Y_{t-1} = Y^* \quad \text{and} \quad R_{t-1} = R^*$$

then

$$Y_t = Y^* \quad \text{and} \quad R_t = R^*$$

Taking equation (1) this implies

$$Y^* = k + bY^* + aR^* \qquad (3)$$

Taking equation (2) in implies

$$R^* = 1 + gR^* + mY^* \qquad (4)$$

Equations (3) and (4) can now be solved using any of the methods described in Chapter 3.

If we let $k = 400$, $l = 200$, $a = 0.1$, $b = 0.8$, $m = 0.1$ and $g = 0.7$, equation (3) becomes

$$Y^* = 400 + 0.8\ Y^* + 0.1^*$$

or $$0.2\ Y^* - 0.1\ R^* = 400 \qquad (5)$$

Equations (4) becomes

$$R^* = 200 + 0.7\ R^* + 0.1Y\ ^*$$

or $$-\ 0.1\ Y^* + 0.3\ R^* = 200 \qquad (6)$$

Multiplying equation (6) by 2 and adding to equation (5) gives

$$0.5\ R^* = 800 \quad \text{or } R^* = 1600$$

Inserting this value into equation (5) gives

$$0.2\ Y^* = 560 \text{ or } Y^* = 2800$$

Solving For the Deviations

Once the equilibrium is found the problem is to discover whether actual incomes i.e. Y_t and R approach it and if sections fast. If Y ¹ Y* and we let m, equal the deviation then

$$m_t = Y_t - Y^* \qquad \text{or} \qquad Y_t = Y^* + m_1$$

If R_1 ¹ R* and we let n_1 equal the deviation, then

$$n_1 = R_t - R^* \qquad \text{or} \qquad R_t = R^* + n_1$$

To find the time path of Y_t and R_t it is necessary to solve for m_t and n_t since Y* and R* are constant with respect to time. Substituting the value for Y_t in terms of m_t and R_t in terms of n_t into equation (1) gives

$$Y^*+ m_1 = k + b(Y^* + m_{t-1}) + a(R^* + n_{t-1}) \qquad (7)$$

Subtracting the same values into equation (2) gives

$$m_1 = bm_{t-1} + an_{t-1} \qquad (8)$$

Substituting the same values into equation (2) gives

$$R^* + n_t = 1 + g(R^* + n_{t-1}) + .00\,(Y^* + m_{t-1}) \quad (9)$$

Subtracting equation (4) from equation (9) gives

$$n_1 = gn_{t-1} + .00m_{t-1} \quad (10)$$

From equations (8) and (10) we can obtain a second-order difference equation in one variable, i.e. etither in n or m.

Second-order Difference Equation in m

$$n_{i-1} = \frac{1}{a}m_t - \frac{1}{a}m_{t-1}$$

From equation (8). This holds for all t, *e.g.*

$$n_4 = \frac{1}{a}m_5 - \frac{1}{a}m_4$$

or $$n_t = \frac{1}{a}m_{t+1} - \frac{b}{a}m_t$$

Inserting these values for n_1 and n_{t-1} into equation (10) gives an equation in one variable , m, in three time periods,

$$\frac{1}{a}m_{t+1} - \frac{b}{a}m_1 = g\left[\frac{1}{a}m_t - \frac{b}{a}m_{t-1}\right] + m m_{1-1}$$

$$= \frac{g}{a}m_t - \frac{bg}{a}m_{t-1} + m m_{t-1}$$

thus $$\frac{1}{a}m_{t+1} - \left(\frac{b+g}{a}\right)m_1 + \left(\frac{ba - am}{a}\right)m_{t-1} = 0$$

Multiplying through by α gives

$$m_{t+1} - (b+g)m_t + (bg = am)m_{t-1} = 0$$

This is true for all t, $e.g.$ $-m_6-(b+g)m_5+(b\alpha-a\mu)m_4 = 0$

thus

$$m_1 - (b + g)m1_{t-1} + bg - amu)_{t-1} = 0$$

This is a second-order difference equation in one variable which can be solved by letting $m_t = m_0 r^t$. If $m_t = m_0 r^1$, then

$$m_0 r^1 - (b + g)\, m_0 r^{t-1} + (bg - m) m_0 r^{1-2} = 0$$

Dividing through by $m_0 r^{t-2}$ gives the auxiliary equation

$$r^2 - (b + g)\, r + bg - a^{.00} = 0 \qquad (11)$$

If $(b + g)^2 > 4(bg - a^{.00})$ this equation will have two distinct real roots. As *a* and .00 are both positive this must be the case in the present example. The deviation will take the form

$$m_1 = a r_1^t + b r_2^t$$

where □ and □ are weights to be determined. To fine □ and □ it is necessary to know the value of income in an initial period in both countries, say Y_0 and R_0. From this *t* is possible to fine m_0 and n_0. With this information on m_0 it is possible to find m_t ⪡ n_1

$$m_1 = bm_0 + an_0$$

From equation (8) and

$$n_1 = gn_0 + {}^{.00}m_0$$

from equation (10). In this case information on m_0 and m_1 is all that is required to solve for □ and □.

The deviation m_1 ® 0 as *t* ® ¥ only if both roots are less than unity in absolute magnitude. When this is so Y_t ® Y* or the tiem path of income is stable in country 1. One negative root with absolute value > 1 will give rise to increasing oscillations and render the system unstable,

If $(b + g)^2 = 4(bg - a^{.00})$ equation (11) will have two equal roots which will be labelled *r,* Actual income in country 1 will approach the equilibrium level Y* if $| r | < 1$.

If $(b + g)^2 < 4\ (bg - a^{\cdot 00})$ equation (11) will have complex roots. This will give rise to oscillations in Y_t. If $bg - a^{\cdot 00} = 1$ the system will have perpetual oscillations of constant amplitude. If $bg - am < 1$ the oscilations will fade away and m_t will tend to zero as $t \to \infty$, i.e. $Y_t \to Y^*$. If $bg - a^{\cdot 00} > 1$ the system will have increasing oscillations so that actual income in country 1 will never approach the equilibrium level Y*.

From equation (8) we can see that if $m_1 \to 0$ as $t \to \infty$, $bm_{t-1} \to 0$ and $am_{t-1} \to 0$. As a is a constant $n_{t-1} \to 0$, i.e. $n_1 \to 0$ as $t_1 \to \infty$ and $R_1 \to R^*$.

Alternatively we can examine the time path of income in country 2 by solving for the deviation n_t. From equations (8) and (10) it is possible to second-order equation in n. Thus it give the exact same equation considered above

Second-order Difference Equation in n

$$m_{t-1} = \frac{1}{m} n_t - \frac{g}{m} n_{t-1}$$

From equation (10). Since this holds for all t then

$$m_t = \frac{1}{m} n_{t+1} - \frac{g}{m} n_t$$

Inserting this values for m_t and m_{t-1} into equation (8) gives

$$\frac{1}{m} n_{t+1} - \frac{g}{m} n_t = \left(\frac{1}{m} n_t - \frac{g}{m} n_{t-1} \right) + a n_{t-1}$$

so
$$\frac{1}{m} n_{t+1} - \left(\frac{b + g}{m} \right) n_t + \left(\frac{bg - am}{m} \right) n_{t-1} = 0$$

Multiplying through by $\cdot 00$ gives

$$n_{t-1} - (b + g) n_t + (bg - am) n_{t-1} = 0$$

or
$$n_t - (b + g) n_{t-1} + (bg - am) n_{t-2} = 0$$

This will give us equation (11). However, the value of the weights □ and □, which do not affect the stability of the system, will depend upon n_0 and n_1 not m_0 and m_1. Hence the value of the deviation n_t may differ frrom m_t for and given *t,* i.e. n_3 need not equal m_3. As mentioned earlier, it is possible to find n_1 from equation (10),. given m_0 and n_0, i.e. given Y_0 and R_0.

14.5 CAPITAL STOCK ADJUSTMENT THEORY OF INVESTMENT

It was assumed that investment is determined by a simple accelerator model. In such a model investment is a function of one variable, income. This was convenient for difference equations in one variable. The capital stock adjustment model assumes investment is a function of two variables, income and capital stock. Investment is assumed to be proportional to the difference between desired capital stock K* this perriod and actual capital stock of the previous period. K_t^* is a function of Y_t as gives the most appropriate level of capital needed to produce an output of V_t, i.e.

$$K_t^* = aY_t$$

where $a > 0$. It would be desirable to have

It = 2)21⃗□ t - □t-1)

l_t = ⃗2)21 y_t - □t-1)

where □t-1 is total capital end of periods. However, since firms do not know Y*t* when investment in period *t* has to be decided upon, the investment function is assumed to take the form

$$l_t = l(K_{t-1}^* - K_{t-1})$$

i.e. $$l_t = l(aY_{t-1} - K_{t-1})$$

Owing to costs of adjustment and to uncertainty about the optimal capital stock, firms are unlikely to attempt to eliminate the difference between K_{t-1}^* and K_{t-1} completely in one period. Thus l will be < 1 but > 0.

Investment in period t is a net addition to capital stock during that period, i.e.

$$K_t = K_{t-1} + I_t$$

$$= K_{t-1} + \lambda(\alpha Y_{t-1} - K_{t-1})$$

thus

$$K_t = (1-\lambda)K_{t-1} + \alpha\lambda Y_{t-1} \qquad (12)$$

Equation (12) is a first-order difference equation in two variables. Capital stock is a function of Y and K lagged one period.

The national income identity

$$Y_t = C_t + I_t$$

where

$$C_t = C^* + cY_{t-1}$$

and

$$I_t = \lambda(\alpha Y_{t-1} - K_{t-1})$$

gives another difference equation in the same two variables with income a function of Y and K lagged one period

$$Y_t = C^* + cY_{t-1} + \lambda(\alpha Y_{t-1} - K_{t-1})$$

i.e.

$$Y_t = C^* + (c + \alpha\lambda)Y_{t-1} - \lambda K_{t-1} \qquad (13)$$

Using the method given above we can solve for Y_t and Y_t. If there is an equilibrium pair of income Y* and capital stock K*, these levels will continue if both are simultaneously, i.e. If $Y_{t-1} = Y^*$ and $K_{t-1} = K^*$. then $Y_t = Y^*$ and $K_t = K^*$. We can see from equations (12) and (13) that achievement of its equilibrium level by one of the variables, if the other does not achieve equilibrium, will not result in the system staying in equilibrium.

Taking equation (12) this gives

$$K^* = (1- \ l)K^* + al\,Y^*$$

i.e. $2)21K^* = \square 2)21y^*$ or $K^* = \square y^*$

Taking equation (13) we have

$$Y^* = C^* + (c + al)Y^* - l\,K^*$$

$$= C^* + C^* + (c + al)Y^* - al\,Y^*$$

thus

$$Y^* = \frac{C^*}{1- \ c}$$

$$K^* = aY^*$$

thus

$$K^* = \frac{aC^*}{1- \ c}$$

A finite equilibrium level of income and capital stock will exist if c ¹ 1. To determine the time path of income and capital stock let z_t equal the deviation of Y_t from Y^* and u_t equal the deviation of K_t from K^*, i.e.

$$Y_t = Y^* + z_t = \frac{C^*}{1-c} + z_t$$

and

$$K_t = K^* + u_t = \frac{xC^*}{1- \ c} + u_t$$

Equation (12) gives

$$u_t = (1- \ l)u_{t-1} + al\,z_{t-1} \qquad (14)$$

Equation (13) gives

$$z_t = (c + al)z_{t-1} - \ l\,u_{t-1} \qquad (15)$$

From equations (14) and (15) its it possible to obtain a second-order difference equation in one variable.

Second-order Difference Equation in *z*

$$\upsilon_{t-1} = -\frac{1}{\lambda}z_t + \frac{(c+\alpha\lambda)}{\lambda}z_1$$

from equation (15), thus

$$\upsilon_t = -\frac{1}{\lambda}z_{t+1} + \frac{(c+\alpha\lambda)}{\lambda}z_1$$

Substituting these values for υ_t and υ_{t-1} into equation (14) gives

$$\frac{1}{\lambda}z_{t+1} + \frac{(c+\alpha\lambda)}{\lambda}z_t = -\frac{(1-\lambda)}{\lambda}z_t + \frac{(1-\lambda)(c+\alpha\lambda)}{\lambda}z_{t-1} + \alpha\lambda z_{t-1}$$

Multiplying both sides by $-\lambda$ gives

thus

$$z_{t+1}-(c+\alpha\lambda)z_t = (1-\lambda)z_t-(c+\alpha\lambda-c\lambda-\alpha\lambda^2)z_{t-1}-\alpha\lambda^2 z_{t-1}$$

thus

$$z_{t+1}-(c+\alpha\lambda+1-\lambda)z_t+(c+\alpha\lambda-c\lambda)z_{t-1}=0$$

or

$$z_t-(c+\alpha\lambda+1-\lambda)z_{t-1}+(c+\alpha\lambda-c\lambda)z_{t-2}=0$$

The auxiliary equation will take the form

$$r^2 - (c+\alpha\lambda+1-\lambda)r + (c+\alpha\lambda-c\lambda) = 0 \qquad (16)$$

The roots of the system may be real or complex depending upon the values of α, c and λ. If $(c+\alpha\lambda+1-\lambda)^2 \geq 4(c+\alpha\lambda-c\lambda)$, equation (16) will have real roots and actual income will approach the equilibrium level if both roots are less than unity in absolute magnitude. If $(c+\alpha\lambda+1-\lambda)^2 < 4(c+\alpha\lambda-c\lambda)$, equation (16) will have no real roots. This will give rise to oscillations which will be damped if $c+\alpha\lambda-c\lambda < 1$, explosive if $c+\alpha\lambda-c\lambda > 1$ or of constant amplitude if $c+\alpha\lambda-c\lambda = 1$.

We could have worked with the deviation u_t instead of z_t. As with the previous example this would have yielded the same auxiliary equation and the same roots.

To solve completely for z_t and u_t we need to know y_0 and K_0. This will give us values for z_0 and u_0. Substituting these values into equations (14) and (15) will provide us with values for u_1 and z_1 respectively. We can then proceed to find the weights, i.e. a and b of the previous example, which fit these initial conditions. The weights for z_t which will satisfy z_0 and z_1 need not take the same values as those for u_0 and u_1.

14.6 MARKET MODEL WITH STOCKS IN CASE OF DIFFERENCE EQUATIONS

Suppose we have a model where the price of a good is set by dealers who hold stocks of the good. We will assume that these dealers adjust their price upwards when they feel stocks are running low and downwards when stocks are getting high. If the desired or normal level of stocks is S*, where S* takes a certain value, and P is price, then

$$P_t = P_{t-1} - \lambda (S_{t-1} - S^*) \tag{17}$$

where $\lambda > 0$. If stocks at the end of period $t - 1$, S_{t-1}, are greater than the desired level, i.e $S_{t-1} > S^*$, dealers will lower their price in the following period, so that $P_t < P_{t-1}$. If $S_{t-1} < S^*$ then P_t will be greater than P_{t-1}. If $S_{t-1} = S^*$ then P_t will be greater than P_{t-1}. If $S_{t-1} = S^*$ then P_t will equal P_{t-1} and price will be stable. λ is the coefficient of adjustment. Equation (17) is a difference equation in two variables with price a function of P and S lagged one period.

We know that stocks in period t will equal stock at the end of period $t - 1$ plus excess supply in period t. If X_t, excess supply in period t, is an increasing function of

and takes the form

$$X_t =$$

where a and A are positive constants,

$$S_t = S_{t-1} + X_t$$

i.e.

$$S_t = S_{t-1} - A + aP_{t-1} \tag{18}$$

Eqution (18) is another difference equation in two variables with stocks a function of P and S lagged one period.

If $S_t \neq S^*$ and $P_t \neq P^*$ and we let u_t be the deviation of S_t from S* and w_t be the deviation of P_t from P*, equation (17) becomes

$$w_t = w_{t-1} - \lambda u_{t-1} \tag{19}$$

and equation (18) becomes

$$u_t = u_{t-1} + aw_{t-1} \tag{20}$$

From equations (19) and (20) we can obtain a second-order difference equation in u

$$w_{t-1} = \frac{1}{a}u_t - \frac{1}{a}u_{t-1}$$

From equation (20), thus

$$w_t = \frac{1}{a}u_{t-1} - \frac{1}{a}u_t$$

Inserting these values for w_t and w_{t-1} into equation (19) gives

$$\frac{1}{a}u_{t+1} - \frac{1}{a}u_t = \frac{1}{a}u_t - \frac{1}{a}u_{t-1} - \lambda u_{t-1}$$

Multiplying across by a and grouping gives

$$u_{t+1} - 2u_t + (1 + a\lambda)u_{t-1} = 0$$

or

$$u_t = 2u_{t-1} + (1+a\lambda)u_{t-2} = 0$$

Letting $u_t - u_0 r^t$ and dividing through by $u_0 r^{t-2}$ we get the auxiliary equation

$$r^2 = 2r + 1 + a\lambda = 0$$

This equation will have no roots since $(-2)^2 < 4(1 + 4a\lambda)$. Consequently the system will have oscillations. These oscillations will be ever increasing since $+a\lambda > 1$ so that $u_t \to \pm\infty$ as $t \to \infty$ and S_t will never approach S*. In the same way P_t will never approach P*.

As this model must eventually imply negative prices, outputs, or both, it could clearly only describe realify over a limited range, beyond which the linear equations cannot hold.

14.7 MARKET MODEL WITH STOCKS IN CASE OF DIFFERENTIAL EQUATIONS

Suppose we have a model where the rate of change in stocks of a good is directly proportional to excess supply X_t and

$$X_t = -A + aP_t$$

where A and a are positive constant, then

$$\dot{S}_t = \lambda(-A + aP_t)$$

where $\lambda > 0$, i.e.

$$\dot{S}_t = -A\lambda + a\lambda P_t \tag{1}$$

If we assume that dealers adjust their actual price P_t towards a target price P'_t at a speed which is proportional to the gap between P'_t and P_t, then

$$\dot{P}_t = (P'_t - P_t) \quad \text{where} \quad \beta > 0$$

P_t is positive when $P_t¢ > P_t$ and negative when $P_t¢ < P_t$

If the target price is related to stocks so that

$$P_t¢ = M - bS_t$$

where M and *b* are positive constants, then

$$\dot{P}_t = b(M - bS_t - P_t)$$

i.e.

$$\dot{P}_t = Mb - bbS_t - bP_t \qquad (2)$$

Equations (1) and (2) are first-order differential equations in two variables, i.e. S and P. In solving for S_t and P_t equations (1) and (2) will give rise to a second-order differential equation.

If and equilibrium S* and P* are simultaneously achieved then $\dot{S}_t = \dot{P}_t = 0$ at these levels. At equilibrium equation (1) becomes

$$P^* = \frac{A}{a} \qquad (3)$$

and equation

$$Mb - bbS^* - bP^* = 0 \qquad (4)$$

Inserting the value for P* from equation (3) into equation (4) gives

$$Mb - bbS^* - \frac{bA}{a} = 0$$

or

$$S^* = \frac{aM - A}{ab}$$

and

$$\dot{S}_t = \dot{u}_t$$

since $\dot{S}^* = 0$. If $P_t \neq P^*$ and we let w_t be the deviation, then

$$P_t = P^* + w_t = \frac{A}{a} + wt$$

and $$\dot{P}_t = \dot{w}_t$$

since P* = 0. Substituting these values for $S_t, \dot{S}_t, P_t$ and $\dot{P}_t$ into equations (1) and (2) gives

and $$\dot{u}_t = a\lambda w_t \quad (5)$$

$$\dot{w}_t = -b\beta u_t - \beta w_t \quad (6)$$

From equations (5) and (6) it is possible to obtain a second-order differential equation in either u or w.

Second-order Differential Equation in u

$$w_t = \frac{1}{a\lambda}\dot{u}_t$$

from equation (5), thus

$$\dot{w}_t = \frac{d}{dt}\left(\frac{1}{a\lambda}\dot{u}_t\right) = \frac{1}{a\lambda}\ddot{u}_t,$$

where $\ddot{u}_t = d\dot{u}_t / dt$. Substituting these values for w_t and $\dot{w}_t$ into equation (6) gives

$$\frac{1}{a\lambda}\ddot{u} = -b\beta u_t - \frac{\beta}{a\beta}\dot{u}_t$$

Multiplying both sides by $a\lambda$ gives

$$\ddot{u}_t + \beta\dot{u} + ab\beta\lambda u_t = 0 \quad (7)$$

path of S_t and P_t we must solve for u_t.

If u_t ® 0 as t ®¥ then S_t ® S*. When u_t ® 0 then $\dot{u}_t$ ® 0 from equation (5) $a l w_t$ or w_t ® 0 since al is constant. Consequently P_t ® P* when u_t ® 0 .

We must find a value for u_t which satisfies equation (7). If we let $u_t = u_0 e^{rt}$, where $u_0 = S_0 - S^*$ and information on S_0 is available, it is necessary to obtain a value or values for r so that $u_t = u_0 e^{rt}$ will satisfy equation (7). If $u_t = u_0 e^{rt}$, then

$$\dot{u}_t = r u_0 e^{rt}$$

and

$$\ddot{u}_t = r^2 u_0 e^{rt}$$

Inserting these values into equation (7) gives

$$r^2 u_0 e^{rt} + \mathrm{b} r u_0 e^{rt} + ab\mathrm{bl}\, u_0 e^{rt} = 0$$

Dividing through by u_0 e^{er} gives

$$r^2 + \mathrm{b} r + ab\mathrm{b}l = 0 \qquad (8)$$

Auxiliary Equation with Two Distinct Real Roots

If $\mathrm{b}^2 > 4ab\mathrm{b}l$, equation (8) will have two distinct real roots which will be labelled r_1 and r_2. Consequently $u_t = u_0 e^{r1^t}$ and $u_t - u_0 e^{r2^t}$ will satisfy equation (7). As in Chapter 12, equation (7) is also satisfied by

$$u_1 = g_1 e^{r1^t} \text{¸} g_2 e^{r2^t} \qquad (9)$$

where g_1 and g_2 are weights to be determined. If r_1 and r_2 are both negative then u_t ® 0 as t ®¥ e^{r1^t} and e^{r2^t} tend to zero as t ®¥ . The system will be unstable if either root is positive.

To determine the weights g_1 and g_2 we must have information on stocks and price at some initiail time, say S_0 and P_0. With this information it is possible to find a value for u_0 and $\dot{u}_0$ which give us two equations in two unknowns, i.e. g_1 and g_2. From equation (9) we have

$$u_0 = g_1 e^0 + g_2 e^0$$

or

$$u_0 = g_1 + g_2 \qquad (10)$$

From equation (9) we also have

$$\dot{u}_t = r_1 g_1\ e^{r2^t} + r_2 g_2 e^{r2^t}$$

thus

$$\dot{u}_0 = r_1 g_1\ e^0 + r_2 g_2 e^0 \qquad (11)$$

i.e. $\dot{u}_0 = r_1 g_1 + r_2 g_2$

But $u_0 = S_0 - S^*$ and

$$\dot{u}_0 = \dot{S}_0 = -\ Al + al\,P_0$$

from equation (1) u_0 and $\dot{u}_0$ are therefore given if S_0 and P_0 are given. It is then solving equation and (11) to find

We could have examined the time path of price by solving for the deviation w_t. This will give us the same auxiliary equation, i.e. equation (8) and hence the same roots, but the weights g_1 and g_2 need not take the same values since u_0 need not equal w_0 and $\dot{u}_0$ need not equal $\dot{w}_0$.

Specific Example

Let l = 1, A= 68, a = 0.4, b = 2, M = 250, b = 0.8, P_0 = 30 and S_0 = 120. Equilibrium S* is given by

$$S^* = \frac{0.4(250) - 68}{0.4(-\ 0.8)} = 100$$

Equation (8) becomes

$$r^2 + 2r + 0.64 = 0$$

i.e.

$$r = \frac{-2 \pm (4 - 2.56)}{2}$$

or

$$r_1 = -1.6 \text{ and } r_2 = -0.4$$

thus

$$u_t = g_1 c^{-1.6t} + g_2 e^{-0.4t}$$

i.e.

$$u_0 = g_1 + g_2 \qquad (12)$$

and

$$\dot{u}_1 = -1.6 g_1 e^{-1.6t} - 0.4 g_2 e^{0.4t}$$

i.e.

$$\dot{u}_0 = -1.6 g_1 - 0.4 g_2 \qquad (13)$$

$$\dot{u}_0 = S_0 - S^* = 120 - 100 = 20$$

$$\dot{u}_0 = \dot{S}_0 = -\text{Al} + a / P_0 = -68 + 12 = -56$$

From equation (12)

$$g_1 + g_2 = 20 \qquad (14)$$

From equation (13)

$$-1.6 g_1 - 0.4 g_2 = -56 \qquad (15)$$

Multiplying equation (14) by 0.4 and adding to equation (15) gives

$$-1.2 g_1 = -48 \qquad \text{or} \qquad g_1 = 40$$

and

$$g_2 = -20$$

from equation (14), thus

$$u_t = 40\,e^{-1.6t} - 20e^{0.4t}$$

$$S_t = 100 + 40e - 20e^{0.4t}$$

Since r_1 and r_2 are both negative then $u_t \rightarrow 0$ as $t \rightarrow \infty$ and $S_t \rightarrow S^* = 100$.

Auxiliary Equation with Two Equal Roots

If $\beta^2 = 4ab\beta\lambda$, equation (8) will have two equal roots i.e. $r_1 = r_2 = r$. As in Chapter 12, equation (7) will therefore be satisfied by

$$u_1 = g_1\,e^{rt} + g_2 t\,e^{rt} \tag{16}$$

thus

$$\dot{u}_t = rg_1\,e^{rt} + g_2\,e^{rt} + rg_2 t\,e^{rt}$$

i.e.

$$\dot{u}_t = rg_1\,e^{rt} + g_2\,e^{rt}(1 + rt) \tag{17}$$

To obtain the derivative of $g_2 t\,e^{rt}$ with respect to t we use the product rule,

i.e.

$$\frac{d(g_2 t\,e^{rt})}{dt} = g_2\,e^{rt}\frac{dt}{dt} + t\frac{d(g_2 e^{rt})}{dt}$$

$$= g_2\,e^{rt} + trg_2\,e^{rt}$$

$$u_0 = g_1\,e^0 + g_2(0)e^0 = g_1$$

from equation (16), and

$$\dot{u}_0 = rg_1\,e^0 + g_2\,e^0(1 + 0) = rg_1 + g_2$$

from equation (17). Given S_0 and P_0 we can find u_0 and $\dot{u}_0$ and solve for g_1 and g_2. For this system to be stable r must be negative.

Auxiliary Equation with Complex Roots

If $b^2 < 4ab$ bl , equation (8) will have complex roots. If we let $r_1 = m + if$ then $r_2 = m - if$, where $m = -\frac{1}{2}$b and $f = \frac{1}{2}$ ($4ab$bl - b^2), and

$$u_t = g_1 e^{mt+ift} + g_2 e^{mt-ift}$$

or
$$u_t = e^{mt}[g_1 e^{ift} + g_2 e^{-ift}]$$

Clearly e^{mt} ® 0 as t ® ¥ if $m < 0$. Consequently to find how u_t behaves over time we must examine $e^{\pm ift}$.

We saw in Chapter 11 that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Applying this to be e^{ift}. where $ft = x$, gives

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \ldots$$

If we let

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots. = \mathrm{A}$$

and

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots. = \mathrm{B}$$

then

$$e^{ix} = \mathrm{A} + i\mathrm{B}$$

Series A and B will converge, i.e. the sum of the first n terms will come very close to a definite number.

$$\frac{d\mathrm{A}}{dx} = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots = -\mathrm{B}$$

and

$$\frac{d\mathrm{B}}{dx} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \mathrm{A}$$

We can now identify A with Cos x and B with sin x. From Appendix 3 we know that

$$\frac{d(\cos x)}{dx} = -\sin x \text{ and } \frac{d(\sin x)}{dx} = \cos x$$

Also when $x = 0$ A(0) = 1 and B (0) = 0. Thus A = cos x and B = sin x, so

$$e^{ix} = \cos x + i \sin x$$

i.e.

$$e^{ift} = \cos ft + i \sin ft$$

and

$$u_t = e^{mt}[g_1(\cos ft + i \sin ft) + g_2(\cos ft - i \sin ft)]$$

u_t is the deviation of S_t from S* and must therefore be real. It will be real if g_1 and g_2 are complex conjungates. If we let $g_1 = u + in$ and $g_2 = u - in$, then

$$u_t = e^{mt}[(u + in)(\cos ft + i \sin ft) + (u - in)(\cos ft - i \sin ft)$$

i.e.

$$u_t = 2e^{mt}(u \cos ft - n \sin ft) \tag{18}$$

or

$$u_t = 2\, e^{mt} \sqrt{(u^2 + n^2)} \left[\frac{u}{\sqrt{(u^2 + n^2)}} \cos ft - \frac{n}{\sqrt{(u^2 + n^2)}} \sin ft \right]$$

We can always find an angle 6)39 such that

thus

$$u_t = 2\,e^{mt}\sqrt{(u^2 + n^2)}(\cos\theta\cos ft - \sin\theta\sin ft)$$

$$= 2\,e^{mt}\sqrt{(u^2 + n^2)}[\cos(\theta + ft)]$$

see Appendix 3. We saw in Chapter 13 that the Cosine of an angle varies from + 1 to – 1 every 360°. Consequently cos $(\theta + ft)$. oscillates between constant limits. $2\sqrt{(u^2 + n^2)}$ will not change over time so that this will not affect the stability of the system. Whether the system will have damped, increasing or constant oscillations will depend on e^{mt}.

If $m < 0$ then $e^{mt} \to 0$ as $t \to \infty$ and $u_t \to 0$. The oscillations due to cos $(\theta + ft)$ will fade away and $S_t \to S^*$.

If $m > 0$ the system will have increasing oscillations as e^{mt} increases with t.

Finding Values for the Weights g_1 and g_2.

To solve for u and n and hence for g_1 and g_2 we use u_0. cos 0 = 1 and sin 0 = 0, thus

$$u_0 = 2\,e^0(u\cos 0 - n\sin 0) - 2u$$

From equation (18), i.e. $u = 0.5u_0$ (19)

We have now solved for u since u_0 is given once we kinow S_0 and S^*. Using the product rule we can obtain

$$\dot{u}_1 = 2\,e^{mt}[-fu\sin ft - fn\cos ft] + 2m\,e^{mt}[u\cos ft - n\sin ft] \qquad ...(20)$$

$d(\cos ft)/dt = -f\sin ft$ because $d(\cos x)/dx = -\sin x$, and if we let $x = ft$, then the function of a function rule gives

$$\frac{d(\cos x)}{dt} = \frac{d(\cos x)}{dx}.\frac{dx}{dt} = -(\sin x).f = -f\sin ft$$

In the same way it can be shown that $d(\sin ft)/dt = f\cos ft$. From equation (20),

$$= 2\,e^0(-fu\sin 0 - fn\cos 0) + 2m\,e^0(u\cos 0 - n\sin 0)$$

i.e. $$\dot{u}_0 = -2fn + 2m\upsilon = -2fn + u_0 m$$

or $$n = \frac{u_0 m - \dot{u}_0}{2f}$$

We have now solved for n since we know the value of m and f and can find u_0 and $\dot{u}_0$ is given S_0 and P_0.

Specific Example

Let $\lambda = 1, A = 10, a = 1, \beta = 1.6, \mu = 20, b = 0.5, P_0 = 14$ and 40 Equilibrium S* is given by

1(0.5)

Equation (8) $$r^2 + 1.6r + 0.8 = 0$$

i.e. $$r = -0.8 \pm i(0.4)$$

i.e. $$r_1 = -0.8 + i(0.4) \text{ and } r_2 = -0.8 - i(0.4)$$

$$m = -0.8 \qquad \text{and } f = 0.4$$

Thus $$u_1 = 2e^{-0.8t}\sqrt{(\upsilon^2 + n^2)}[\cos(\theta + 0.4t)] \qquad (22)$$

$e^{-0.8t} \to 0$ as $t \to \infty$ so that $u_t \to 0$ as $t \to \infty$. Consequently the system is stable. To fined values for υ and n we use u_0 and $\dot{u}_0$.

$$u_0 = S_0 - S^* = 40 - 20 = 20$$

$$\dot{u}_0 = -A\lambda + a\lambda P_0 = -10 + 14 = 4$$

But $\upsilon = 0.5u_0 = 10$ from equation (19) and

$$n = \frac{20(-0.8) - 4}{2(0.4)} = -25$$

from equation (21). $\sqrt{(\upsilon^2 + n^2)} = \sqrt{725}$ and the angle θ whose cos is $10/\sqrt{725}$ is 68° approximately, thus

$$u_t = 10\sqrt{29}\, e^{0.8t}[\cos(68° + 0.4t)]$$

from equation (22), and

$$S_t = 20 + 10\,\ddot{O}29\, e^{0.8t}[\cos(68° + 0.4t)]$$

14.8 NATIONAL INCOME MODEL

Suppose we have a closed model without government activity where consumption is a function of 'normal' income W_t, i.e.

$$C_t = A + cWt$$

and investment is assumed to be proportional to the difference between desired capital stock

$$K_t^* = I_t = l(K_t^* - K_t) = l(uW_t - K_t)$$

or

$$K_t^* = ul\,W_t - l\,K_t \qquad (23)$$

where l and u are positive constants and $K_t^* = uW_t$.

If normal income W_t adjusts towards actual income y_t at a speed which is proportional to the gap between W_t and y_t , i.e.

$$W_t = \beta\ (y_t - W_t)$$

Where $\beta > 0$

$$= \beta\ (C_t + I_t - W_t)$$

Inserting the above values for C_t and I_t gives

$$W_t = \beta[A +_c W_t + V\lambda W_t - \lambda K_t - W_t] \qquad (24)$$

Equations (23) and (24) are first order differential equations in two variables, i.e W and K. We can solve for W_t and K_t in the same way as we did for S_t and P_t.

14.9 SUMMARY

We end this lesson by summarising what we have covered in it :

i) Cocweb Model

ii) Foreign Trade Multiplier Model

iii) Capital Stock Adjustment

iv) Market Model with stocks; and

v) National Incomes Model.

14.10 LESSON END EXERCISE

1. Suppose we have a two - country model where Y is income in country 1 and R is income in country 2 In country 1

$$C_i = 10 + 0.9$$

$$I = 40$$

$$M_t = 150 + 0.1\ Y_{t-1}$$

$$X_t = 200 + 0.3\ R_{t-1}$$

In country 2

$$C_t^{¢} = 200 + 0.9\ R_{t-1}$$

$$I_t^{¢} = 50$$

$$M_t^{¢} = 200 + 0.3\ R_{t-1}$$

$$X_t^{¢} = 150 + 0.1\ Y_{t-1}$$

Comment on the stability of this system. If $Y_0 = 2030$ and $R_0 = 1090$, find Y_t, R_t, Y_4 and R_4.

2. Given that the relation between the price of a good and stocks of that good held by dealers is

$$P_t = P_{t-1} - 0.8\ (S_{t-1} - 120)$$

where 120 is the desired level of stock, and that excess supply, X_t, of this good in period t is a function oif P_{t-1}, i.e.

$$X_t = -\ 52 + 0.2\ P_{t-1}$$

Excess supply is added to stocks so that

$$S_t = S_{t-1} + X_t$$

Comment on the stability of the system and find S_t ancd S_2 when $S_0 = 180$

and $P_0 = 100$.

3. Given a closed system without government activity where

$$C_t = 100 + 0.8\ Y_{t-1}$$

$$I_t = 0.5\ (K^*_{t-1} - K_{t-1})$$

where K^*_{t-1} is desired capital stock in period $t - 1$ and

$$K^*_t = 2Y_t$$

Is this system stable ? Find Y_t and Y_6 when $Y_0 = 520$ and $K_0 = 1008$.

14.11 SUGGESTED READINGS

Black. J. and Bradley. J. F. Essential Mathematics for Economists.

Henderson J. M. and Quandt. R. E. Microeconomic Theory : A Mathematical Approach.

Aggarwal, C.S & R.C Joshi : Mathematics for students of Economics (New Academic Publishing Co.)

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M.A. ECONOMICS		LESSON NO. 15
COURSE NO. 103	SEMESTER IST	UNIT–IV

LINEAR PROGRAMING

Structure

15.1 INTRODUCTION

In the II world war, the origin of linear programming took place for nobilisation of war programming took place for mobilisation of war programme. The problem of allocating parts, components finished goods, all in proper quality and quantity, among the huge complex of machinery, so as to attain an over all war tooting production and it seemed that human mind was incapable of aliminating these difficulties. The tool whicvh would solve complex of problem is known as linear programming.

15.2 OBJECTIVES

After reading this unit you should be able to apply graphs on mathematical models. As this unit made you able to recognize :

- Linear Programming Application
- Mathematical form of Linear programming
- Graphical Methods of solving LPP.
- Primal and Duality

15.3 LINEAR PROGRAMMING DEFINITIONS

15.3.1 Mathematical Form of Linear Programming

Ans. Linear programming is a tool (or technique) which is used in decision making in business for obtaining maximum and minimum values of quantities subject to certain constraints.

We deal the problem of allocating limited resources in linear programming among competing activities in as optimal. Thus, mathematically problem of linear programing may be stated as one of maximizing a linear objective function of the following form :

$$f = c_1x_1 + c_2x_2 + ...c_nx_n.$$

Subject to linear constraints of the form.

$$a_{11}x_1 + a_{12}\, x_2 + ... + a_{n1}\, x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}\, x_n \ge b_2$$

....

....

$am_1x_1 + am_2x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \le bn$

and non-negatively constraints are,

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \ \ldots\ldots\ldots\ldots \ x_n \geq, \ 0$

In, matrix notation, we can write above problem as

Max (or min) = C^TX where $C = (c_1, c_2, \ldots\ldots, c_n)$,

$$X = (x_1, x_2, \ldots\ldots, x_n)T$$

Subject to constraint $\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\ am_1 & am_2 & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ x_n \end{bmatrix} \geq \le \begin{bmatrix} b_1 \\ b_2 \\ \cdot\cdot \\ bm_1 \end{bmatrix}$

($i.e.$) $AX \geq B$.

where (i) $x_1\ x_2, \ldots\ x_n$ are decision (or choice) variable $c_1, c_2 \ldots\ldots, c_n$ are called cost or profit co-efficient

(iii) aij, ($i = 1, 2, \ldots, m$) $j = 1, 2 \ldots, n$ are known as structural co-efficient ($i.e.$) exchange coefficient of the *jth* decision variables in the *ith* constraints.

(iv) $b_1, b_2, \ldots, bm$ denote requirement or availability of in constraints and the expression ($\le$, $\geq$) shows that each constraint may take only one of three possible forms (a) $\le$ (b) = (c).

The restrictions ($xj \geq$ '$0, j = 1, 2, \ldots, n$) means that the *xj's* must be non-negative.

The above constraint optimization problem may have

(i) no feasible solution, *i.e.* there may not exist values $xj,\ j = 1, 2, \ldots., n$ that satisfy every restriction or constraints.

or (ii) a unique optimal feasible solution.

or, (iii) more than one optimal feasible solution.

or, (iv), a feasible solution for which the objective function is unbounded.

15.3.2 Assumptions of Linear Programming

(i) Linearity. The linear programming problems are always expressed in linear relations. It means that the objective function and structrucal constraints are linear.

(ii) Divisibility. In these problems fractional or integral values are permitted to use. It implies that we can use any quantity of inputs and produce any quantity of various inputs and the quantities not necessarily being complete with.

(iii) Constant Prices. In the operations of linear programme prices of inputs and outputs are considered to be constant during the whole operation. In other word prices are not variable factors in linear programming problems.

(iv) Presence of constraints in the activities. It means that there are only a finite number of outputs possible, and a finite number of input requirements. It means only finite quantities can be produced and finite number of inputs can be used. Thus only those problem which contain some kinds of constraints in their activities can be solved with the help of linear programming.

15.3.3 Objective function, structural constraint, non-negativity constraints, basic solutions, basic feasible solutions, optimal feasible solution and degenerate dilution.

Ans. Objective Function. The function which is required to be maximized or minimized is called objective function.

Structural Constraints. The quantities which state the side conditions on the different activities of the problem are called structural constraints.

Non-negativity Constraints. The non-negativity constraints are those which assume that there can be no negative values of the variables involved in the problem.

e.g. Maximise $f = x + 4y$ ...(1)

subject $2x + 3y \le 4$ ...(2)

$3x + y \le 3$ ...(3)

and $x \ge 0, y \ge 0$ ...(4)

So, from above

1 is the objective function

2 and 3 are structural constraints which tell that capacity in the departments is restricted to a given constraints.

4. is the non-negativity constraints, i.e., variables can be atmost be zero but can never be negative.

Basic Solutions. When we have a set of m simultaneous equations in 'n' unknown ($n > m$), a solution obtained by setting ($n - m$) of the variables equal to zero and solving the remaining m equations in m unknown is called a basic solution, zero variables ($n - m$) are known as non-basic variables and remaining are called basic variable and constitute a basic solutions.

Basic Feasible Solution. A feasible solution to a general □. *p* problem, which is also basic solution, is called a basic feasible solution.

Optimal Feasible Solution. An optimal feasible solution is any basic solution, which optimize (maximize or minimize) the objective function of a general linear programming problem.

Degenerate Solution. A degenerate solution is a basic solution to the system of equations, if one or more of the basic variables become equal to zero.

15.4 FORMATION OF LINEAR PROGRAMMING PROBLEMS

Following steps are involved in the formulations.

Step I. First determine the key decision to be made from the study of the problem.

Step II. Now we identify the variables and assume symbols x_1, x_2, ... for variable quantities studied in Step I.

Step III. Next we express the possible alternatives mathematically in terms of variables. Generally the set of feasible alternatives in the given problem is $\{(x_1, x_2, x_1 > x_2 > 0\}$

Step IV. Express the objective quantitatively and express it as a linear function and variables.

Step V. Lastly, we express the constraints as linear equation or inequalities in terms of variables.

Q. Prima Enterprises manufacture three types of clothes.

The Boy requires half metre of red cloth, $1\frac{1}{2}$ metres of green cloth metres $1\frac{1}{2}$ of black cloth and 5 Kg. of fibre. The Girl requires $\frac{1}{2}$ metre of red cloth, 2 metres of green cloth, 1 metre of black cloth and 6 kg of fibre. The 'Dog' requires $\frac{1}{2}$ metre of red, 1 metre of green, $\frac{1}{2}$ metre of black and 2 kg of fibre. The profit on the three are respectively 3, 5 and 2 rupees. The firm has 1000 metres of red, 1500 metres of green, 2000 metres of black and 6000 kg of fibre. Find the no. of dolls of each type to be manufactured. Set up a L.P.P for Max. Profit.

Sol. We can summarise the data of given problem in a tabular form :

Decision Variables	Dolls	A Red Cloth	B Green Cloth	C Black Cloth	Doll fibre	Profit per Item
x_1	Boys I	½	1½	1½	5	Rs. 3
x_2	Girls II	½	½	1	6	Rs. 5
x_3	Dogs III	½	1	½	2	Rs. 2
Units available		1000	1500	2000	6000	

Objective Function. Production of one unit of product I yields profit of Rs. 3 (*i.e.*) production of x unit of I, will provide Rs. $3x_1$ equally, production of x_2 units of II and x_3 units of III will provide the total profits $5x_2$ and $2x_3$. The total profits will be

$$Z = 3x_1 + 5x_2 + 2x_3.$$

These profits are to be maximized.

Constraints. There are 4 constraints. Production of one unit of product I, requires $\frac{1}{2}$ metre of red cloth, production of one unit of II requires $\frac{1}{2}$ metre of red cloth and production of one unit of III requires $\frac{1}{2}$ metre of red cloth.

$\therefore$ Production of x_1, x_2 and x_3 units of product I, II and III will require $\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3$ units of red cloth. The firm has only 1000 units of red cloth.

Mathematically the constraint is $\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \leq 1000$

IIly, the constraint of Green cloth is $1\frac{1}{2}x_1 + 2x_2 + x_3 \leq 1500$

and constraint of black cloth is $1\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 \leq 2000$. and constraint of fibre is $5x_1 + 6x_2 + 2x_3 \leq 6000$.

Further we can't have negative production

(*i.e.*) $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

These are called non-negative constraints.

Now above LPP can be stated mathematically

Max $Z = 3x_1 + 5x_2 + 2x_2$, subject to

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \leq 1000,$$

$$1\frac{1}{2}\, x_1 + 2x_2 + x_3 \text{ £ } 1500,$$

$$1\frac{1}{2}\, x_1 + x_2 + \frac{1}{2}\, x_3 \text{ £ } 2000,$$

$$5x_1 + 6x_2 + 2x_3 \text{ £ } 6000,$$

$$x_1, x_2, x_3 \geq 0$$

Q. : Suppose any firm decide to make two products I and II from its available resources. Resources with the firm are 5th units of input A and 560 units of input B. Inputs with the firm can be used. To make either product I or II, or the combination of I and II.The firm knows that to make one unit of product I requires 4 units of input A and 8 units of input B and the production of one unit of product I, provides the firm a profit of Rs. 50. Similarly it is known that the production of one unit of product II, requires 25 units of input A and 16 units of input B yields a profit of Rs.90. The problem is to find, how many units of product I and II should be made by the firm, so that its profit is maximized, remaining with the input constraints.

Sol. Max $f = 50\, x_1 + 90\, x_2$

Subject to constaints $4x_1 + 25x_2 \text{ £ } 500$

$8x_1 + 16x_2 \text{ £ } 560$

and $x_1 > 0, x_2 > 0$

Q. A small scale manufacturer has production facilities for producing two different products. Each of the products requires three different operations, grinding, assembling and testing. Product I requires 15, 20 and 10 minutes for grinding, assembly, and test respectively whereas as product II requires 7.5, 40 and 45 minutes for grinding, assembly and testing. The production run calls for attleast 7.5 hours of grinding time, atleast 20 hr of assembly time and atleast 15 hours of testing time. If product I costs Rs. 60 and product II costs Rs. 90 to manufacture, determine the number of units of each product, the firm should produce, in order to minimise the cost of operation.

Sol. Min Z = $60\, x_1 + 90\, x_2$ subject to

$$15x_1 + 7.5\, x_2 \geq 7.5,$$
$$20x_1 + 40\, x_2 \geq 20$$
$$10x_1 + 45\, x_2 \geq 15$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

15.4.1 Graphical Method of Solving LPP (Linear Programming Problem)

Rules/Steps to slove LPP by graphic method :

(*i*) Formulate the LPP in mathematical form in terms of series of mathematical constraints and an objective function.

(*ii*) Construct the graph for the problem so formulated by drawing constraint line. We write each inequality in the constraint equation as equality. Given any arbitrary value to one variable and get the value of other variable by solving the equation. Similarly give another arbitrary value to the variable and determine the corresponding value of other variable. We plot these two sets of values. Join these points by a straight line. We carried out this exercise for each of the constraint equations. So, we have as many as many straight lines as there are equations, each straight line respresenting one constraint.

(*iii*) Identify feasible region (solution space or convex region) and find the vertices (extreme points) of the feasible region (convex region). We determine the area which satisfies all the constraints simultaneously. For greater than and greater than or equal to constraints the feasible region will be the area lies above the constraint line. For less than and less than or equal to constraints, this area is generally the region below these lines.

(*iv*) Now we select the graphic solution techniques and proceed to solve.

- (*a*) First we identify each of the convex or extreme points of the feasible region either by visual inspection or by the method of simultaneous equations.

- (*b*) Evaluate the value of objective function at these vertices.

≥

≥

(c) Determine the vertex of which objective function attains its maximum or minimum.

(d) Interpret the optimum solution so obtained.

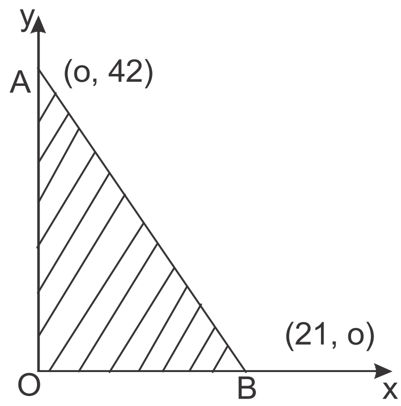
Q. Solve by graph.

Maximize Z = 8*x* + 6*y*,

subject to 6*x* + 3*y* £ 126

2*x* + 4*y* £ 96

and *x*, *y* ³ 0.



Sol. Step I First constraint is

$6x + 3y$ £ 126

when $x = 0$, then $y = 42$ [$\because 6(0) + 3.y = 126$]

In this way, we get first point (A) as (0, 42)

To get second point put $y = 0$

then, $x = 121$ [$\because$ $6x + 3(0) = 126$]

□ Second point (B) is (21, 0)

By joining these we get a straight line AB.

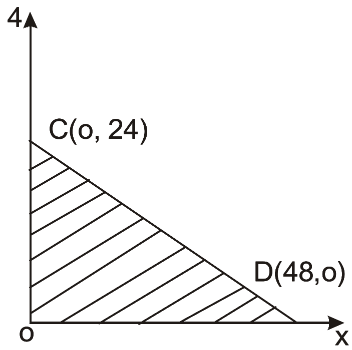
Second Constraint.

When $x = 0$, then y = 24 [$\because$ $2x + 4y = 96$]

□ C (0, 24)

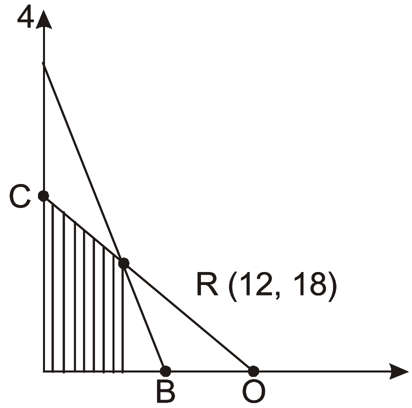
When $y = 0$, then $x = 48$

□ Point D will be (48, 0)



By joining these points, we get a straight line CD.

Step II. After plotting the two constraints, the next step is to find the possible region. The possible region is that region in the graph which satisfies all the constraints simultaneously.



The region CRBO region represents the feasible region.

These extreme points are of signifcance in optimal solution. Optimal solution will always be one of these extreme points.

Point $C(0, 24)$, $x = 0$, $y = 24$

$$Z = 8x + 6y$$

$$= 8 \times 8 + 6 \times 24$$

$$= 144$$

Point B $(21, 0)$, $x = 21$, $y = 0$

$$Z = 8 \times 21 + 6 \times 0$$

$$= 168$$

Point R $(12, 18) = x = 12$, $y = 18$

$$Z = 8\times 12 + 6 \times 18$$

$$= 204$$

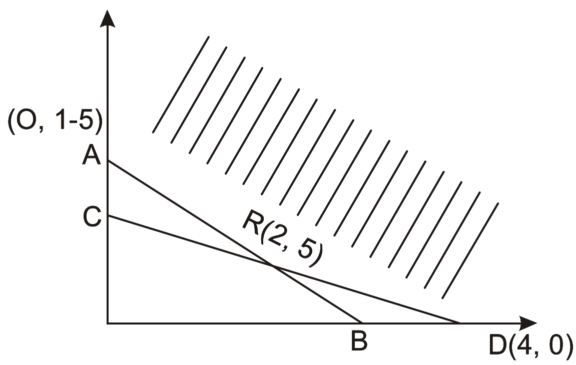
So, combination R(12, 18) is the optimal and 204 is maximum possible level of profit.

Q. Minimize C = $6x + 30y$

Subject to $x + 2y$ ³ 3

$x + 21y$ ³ 4

and, $x \geq 0$, $y \geq 0$



Sol.

A = (0, 1.5)

B = (3, 0)

C = (0, 1)

D = (4, 0)

R = (2,.5)

D(4, 0) gives least cost and ` 24 is the minimum level of cost.

Q. Solve the following LP problem by graphic method.

Mini $2x_1 + 3x_2$

Subject to constraint $x_1 + x_2 \geq 8$,

$3x_1 + 5x_2 \geq 30$

$2x_1 + x_2 \geq 12$

$x_1, x_2 \geq 0$

Sol.

At Î (0, 12)

$Z = 2 \times 0 + 3 \times 12 = 36$

At N (4, 4)

$Z = 2 \times 4 + 3 \times 4 = 20$

At M (5, 3)

$Z = 2 \times 5 + 3 \times 3 = 19$

At D (10, 0)

$Z = 2 \times 10 + 0 = 20$

Z is min at M(15, 3) and min Z = 19

Q. Solve it :

i) Max. $f = 2x + 5y$ subject to

$$x + 4y \le 24$$

$$3x + y \le 21$$

$$x + y \le 9$$

$$x \ge 0,\ y \ge 0$$

i) Maximize $z = 8x + 6y$

subject to $6x + 3y \le 126$

$$2x + 4y \le 96$$

iii) Maximize $f = 2x + 5y$

subject to

$$x + 4y \le 24$$

$$3x + y \le 21$$

$$x + y \le 9$$

$$x \ge 0,\ y \ge 0$$

15.4.2 Simplex Method for Solving LPP (Linear Programming Problem)

Simple method of solving linear programming problems is the most powerful and general method. It was developed by an American Scholar Mr. Dantzig in 1947. It is a repetitive technique and is based on the property that the optimum solution to a linear programming problem. If it exists is always one of the basic feasible solution. The idea of the simplex method is to start with some initial extreme point, compute the value of the objective function and then see whether the latter can be improved upon by moving an adjacent extreme point. For proper understanding of simplex method, let us discuss few definitions.

(a) Standard Form. When all the constraints in a linear programming are written as equalities, it is called standard form. The optimum solution of the standard form of

LPP is the same as the optimum solution of the original formulation of the LPP.

(b) Slack variable when we add a variable to the left hand side of is less than a equal to constraint to convert the constraint into an equality, the variable is called slack variable. The value of the variable can usually be interpreted as the amount of unused resources.

Def. Let the constraints of general LPP be $\sum_{j=1}^{n} aij\ xj \le bi,\ k+1 \le i <$

Then the non-negative variable S_{n+i} which satisfies.

$\sum_{j=1}^{n} aij\ xj + S_{n+i} = bi$ are called slack variables.

(*i.e.*) S_{n+1} is slack variables.

(c) Surplus Variables. When we subtract a variable from the left hand side of 'a greater than or equal to' constraint to convert the constraint into an equality that variable is called surplus variable.

Let the constraints of general LPP be $\sum_{j=1}^{n} aij\ xj \ge bi,\ 1 + k \le i \le$

Then the non-negative variable, S_{n+1}, which satisfies

$\sum_{j=1}^{n} aij\ xj - S_{n+i} = bi$ is called surplus variable.

15.4.3 Simplex Method's Steps to Solve

Max. $a_1x_1 + a_2x_2$ s.t.

$$b_{11}\,x_1 + b_{12}x_3 \le c_1$$

$$b_{21}x_1 + b_{22}x_2 \le c_2$$

$$x_1x_2 \ge 0$$

Step I. Convert inequalities into equations by adding a slack variables.

$$b_{11}x_1 + b_{12}x_2 + s_1 = c_1$$

$$b_{21}x_1 + b_{22}x_2 + s_2 = c_2$$

$$a_1x_1 + a_2x_2 + 0.s_1 + 0.s_2, \text{ where } x_1, x_2, s_1, s_2 \geq 0$$

Step II. The simplex Tabular

	x_1	x_2	s_1	s_2	
	b_{11}	b_{12}	1	0	c_1
	b_{21}	b_{22}	0	1	c_2
z_1	$-a_1$	$-a_2$	0	0	

Choose the smallest of a_1, a_2,

(*i.e.*) min (a_1, a_2)

that column $^{23}_{11}$ smallest no. is the privileged column

Suppose, $-a_2 < -a_1$

then a_2 is a privileged column.

Dividing each constraints by the corresponding element of the pre-column.

$$\frac{c_1}{b_{12}}, \frac{c_2}{b_{22}}$$

If $\frac{c_1}{b_{12}} < \frac{c_2}{b_{22}}$ then b_{12} is pivotal element

last column is column of constraints.

Dividing that each row by b_{12}.

Step III. Find privilege column and pivotal element.

Step IV. Divide the row that contains the pivotal element by pivotal element.

Step V. Transfers Row and find the value.

Q. Maximize $z = 20x_1 + 25 x_2$

subject to

$12x_1 + 16 x_2 £ 100$

$16x_1 + 8x_2 £ 80$

$x_1, x_2 \geq 0$

Solve LPP by using simple method.

Sol. Since the LPP is of maximization form, so we use slack variables s_1, and s_2 for each constraint and equate them to zero.

 $12x_1 + 16x_2 + s_1 = 100$

$16x_1 + 8x_2 + s_2 = 80$

our objective function is -

$z = 20x_1 + 25x_2 + 0s_1 + 0s_2$

Let's construct a simple table for LPP

	c_j	Basic	Solution	20	25	0	0	Min Ratio
	c_B	variables	variables	x_1	x_2	s_1	s_2	
R_1	0	s_1	100	12	(16)	1	0	$\frac{100}{16} = 6.25$
R_2	0	s_2	80	16	8	0	1	$\frac{80}{8} = 10$
		z_j		0	0	0	0	
		$z_j - c_j$		-20	-25	0	0	

Since -25 is least, so column corresponding to -25 becomes our key column.

To find the minimum ratio, divide the solution variable by the respective element of the key column.

The value which is minimum in min. Ratio is known as min. ratio and the row corresponding to it is key row.

By intersection of key row and key column, you will get a key element, i.e., 16.

Now, we will construct interaction - I. Since key element belongs to column x^2 and row s[1].

So introducing x_2 and replacing s_1, in table - 2

	c_j	Basic	Solution	20	25	0	0	Min Ratio
	c_B	variables	variables	x_1	x_2	s_1	s_2	
R_1	25	x_2	100/16 = 25/4	3/4	1	1/16	0	$\frac{25}{3}$
R_2	0	s_2	30	(10)	0	-1/2	1	3
		z_j		75/4	25	25/16	0	
		$z_j - c_j$		-5/4	0	25/16	0	

Now, apply the operation on R'_2 as

$$R'_1 = R2 - 8R'_1$$

Since, $-\frac{5}{4}$ in the $z_j - c_j$ is minimum, so we took the column corresponding to this value as key column. Then, dividing the solution variable by the key column to get the minimum value and corresponding to the minimum value and corresponding to the minimum ratio, there will be a key row. Again, the intersection will provide a key element, i.e. 10.

Similarly, do the interaction - II by introducing x_1 and replacing s_2. In table 3.

Repeat this process, until $z_j - c_j$ becomes zero.

c_j	Basic	Solution	20	25	0	0
c_B	variables	variables	x_1	x_2	s_1	s_2
25	x_2	4	0	1	1/10	-3/40
20	x_1	3	1	0	-1/20	1/10
	z_j	160	20	25	1.5	18.12
	$z_j - c_j$		0	0	1.5	18.12

As all $z_j - c_j$ are positive, so we stop here, as $z_j - c_j$ are also '0' for x_1 and x_2.

□, Basic variable and solution variable gives the value of $x_1 = 3$ and $x_2 = 4$.

and, max. z as 160.

Check : $z = 20(x_1) + 25(x_2) + 0s_1 + 0s_2$

$= 20(3) + 25(4)$

$= 60 + 100$

$= 160$

□, z is maximum at $x_1 = 3$ and $x_2 = 4$, with the value of 160.

Solve it :

i) Max $z = 4x_1 + 6x_2$

subject to constraints

$x_1 + 3x_2 \leq 240$

$3x_1 + 4x_2 \leq 370$

$2x_1 + x_2 \leq 180$

$x_1, x_2 \geq 0$

ii) Max $z = 5x + 8y$

subject to constraints

$x + 2y \leq 8$

$3x + 4y \leq 20$

$x, y \geq 0$

iii) Min $8u + 20u$ st $4 + 3v$ ³ 5

$2u + 4v$ ³ 8

4.v > 0

iv) Min $8u + 10u$ st $24 + v$ ³ 1

$u + 2u$ ³ 1

Note. In that cases first of all find dual of the function and then find its values through simplex method that are maximising value. In order to find the minimising values, the value of $s_1 = x$ and $s_2 = y$

□ Consider and but these values in M in function to get min value.

15.5 PRIMAL AND QUALITY

In the linear programming we are to maximise or minimize, the given function. The fact every linhear programming problemhas a counter part called its dual. The given problem which to be maximised or minimized maximized or minimized is called primal problem. If the primal problem. If the primal problem is one of maximizing the function then dual problem is one of minimizing the function. Similarly, if the primal is a minimization problem. Similarly, if the primal is a minimization problem, then the dual is a maximization problem. Every primal problem can be transferred into dual and every dual problem can be transferred into primal problem.

PRIMAL AND DUALITY

Relationship. The primal and dual problems can be derived from each other and we have a unique dual (primal) associated with the primal (dual).

Consider the general linear programing problems,

Maximize Z = $c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

subject to $\quad a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}\,x_n \le b_1$

$$a_{21}\,x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}\,x_{2n} \le b_2$$

$$am_1x_1 + am_2x_2 + \ldots + a_{mn}\,x_n \le b_n$$

$$x_1, x_2, \ldots., x_n \ge 0$$

It is a primal problem. The dual of the above problem is

Minimize z = $b_1y_1 + b_2y_2 + \ldots. + b_my_m$, subject to

$a_{11}y_1 + a_{21}y_2 + \ldots. + a_{m1}y_m \ge c_1$

$a_{12}y_1 + a_{22}y_2 + \; ..... + am_2\, y_m \geq \; c_2$

....

$a_{1n}y_1 + a_{2n}y_2 + \; .... + a_{mn}\, y_m \geq \; c_n$ and

$y_1 y_2$ ...., $y_m \geq 0$.

Y_1, Y_2 .... , y_m are the dual decision variables

Q. Max. Z = 45 x_1 + 80x_2 subject to

$5x_1 + 20x_2 \leq 400$

$10\, x_1 + 15x_2 \leq 450$, x_1, $x_2 \geq 0$

Sol. The dual of this problem is Min f 400 y_1 + 450y_2

subject to $5y_1 + 10y_2 \geq 45$

$20\, y_1 + 15y_2 \geq 80$, $y_1, y_2 \geq 0$

Q. Minimize Z = $3x_1 - 2x_2 + 4x_3$, subject to

$3x_1 + 5x_2 + 4x_3 \geq 7$,

$6x_1 + x_2 + 3x_3 \geq 4$

$7x_1 - 2x_2 - x_3 \geq 10$

$x_1 - 2x_2 + 5x_3 \geq 3$,

$4x_1 + 7x_2 - 2x_3 \geq 2$, $x_1, x_2, x_3 \geq 0$

Sol. The dual to given primal problem is :

Maximize $f = 7y_1 + 4y_2 + 10y_3 + 3y_4 + 2y_5$

subject to, $3y_1 + 6y_2 + 7y_3 + y_4 + 4y_5 \leq 3$

$5y_1 + y_2 - 2y_3 - 2y_4 + 7y_5 \leq -2$

$4y_1 + 3y_2 - y_3 + 5y_4 - 2y_5 \leq 2$,

$y_1, y_2, y_3, y_{41}\; y_5 \geq 0$

15.5.1 Basic Theorems of LPP

Dual of the dual is a primal

Dual Formation Procedure

1. If the primal problem is of minimization, the dual problem is of maximiations and vice-versa.
2. If the primal problem involves $\leq$ signs in the constraints, the dual problem will have $\geq$ signs in its constraints and vice-versa.
3. The coefficients of objective function in the primal problem take the position of constraints and vice-versa.
4. In the constraints inequalities, the coefficient which are found by going from left to right in the primal problem are positioned in the dual problem from top to bottom and vice-versa.
5. A new set of variables appears in the dual.
6. Sign of non-negativity constraints do not change.
7. Neglecting the number of non-negativity constraints, if these are *n* variables and *m* constraints in the primal problem, in the dual, there will be *m* variables and *n* constraints.

15.6 SUMMARY

We end this lesson by summarizing what we have covered in it :

i) Linear Programming Application

ii) Mathematic form of Linear programming.

iii) Graphical methods of solving LPP.

iv) Simplex Methods of solving LPP.

v) Primal and Quality.

15.7 LESSON AND EXERCISE

Q.1 What is linear programming ? What are its underlying assumtions ?

Q.2 Write the steps of solving LPP by simplex method ?.

Q.3 Use simplex method of solve the following LPP.

Max. $Z = 107x_1 + x_2 + 2x_3$, subject to constraint

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5$$

$$3x_1 + x_2 - x_3 \le 0$$

$$x_1, x_2, x_3 \ge 0$$

Q. 4. Solve the following LPP graphically, maximise $Z = 20x_1, + 10x_2$, subject to $x_1 + 2x_2 \le 40,\ 3x_1 + x_2\ 30,\ 4x_1 + 3x_2\ 60,\ x, x \ge 0$

15.8 SUGGESTED READINGS

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INPUT OUTPUT ANALYSIS

Structure

16.1 INTRODUCTION

Input and output analysis is a method of calculating income and employment multiplies which takes account of differences in technology between industries and of the linkages between industries. It analysis seeks to explain how one industry sector affects others in the same nation or region. The analysis illustrates that the output of one sector can it turn become an input for another sector, which results in an interlinked economic system. The analysis is represented as a matrix, where different rows and coloumns are filled with values representing the inputs and outputs of varies sectors.

16.3 INPUT-OUTPUT ANALYSIS

Output analysis developed by the 20th century Russian born U.S economist wassily W. Leontief, in Russian born U.S. economist Wassily W. Leontief, in which the interdependence of an economy's various productive sectors is observe by viewing the product of each industry both as a commodity demanded for final consumption and as a factor in the production of itself and other goods. The analysis usually involves constructing a table in which each horizontal row describes how one industry's total product is divided among various production processes and final consumption. Each vertical column denotes the combination of productive resources used within one industry.

16.4 INPUT-OUTPUT TANSACTION TABLE

Suppose there are three producing sectors in an economy and that the production of each sector is being used as an input in all the sectors and is employed for final consumption.

Let the total output of three sectors be x_1, x_2, x_3 and d_1, d_2, d_3 are the amounts of final demand consumtpion, capital formation and exports for output of these sectors.

Let the amounts of products of sector I employed as an input in Ist, 2nd and 3rd sectors be x_{11}, x_{12} and x_{13} respectively. Similarly let x_{21}, x_{22}, x_{23} be the amount of product of sector II used as an input in Ist, 2nd 3rd sectors respectively etc.

We can translate in the following cary the distribution of total products of 3 producing sectors

Input-Output Transaction Table

Providing Sector No.	Total input of the sector	Input requirement of producing sectors			Requirements for final use
		x_1	x_2	x_3	
1	2	3	4	5	6
1	x_1	x_{11}	x_{12}	x_{13}	d_1
2	x_2	x_{21}	x_{23}	x_{23}	d_2
3	x_3	x_{31}	x_{32}	x_{33}	d_3
		L_1	L_2	L_3	

Primary input (Labour) Total Primary Input (= L)

From the above table, use derive the following two equations :

(i) From above table columns 3, 4, 5 provide the total input (from all sectors) utilised by each sector for its production. That is coloumn 3 provides the production function of sector 1 etc.

$$x_1 = f_1 \ (x_{11,} \ x_{21}, \ x_{31}, \ L_1)$$

$$x_2 = f_2 \ (x_{12}, \ x_{22}, \ x_{32}, \ L_2)$$

$$x_3 = f_3 \ (x_{13}, \ x_{23}, \ L_3) \text{ etc.}$$

16.5 TECHNOLOGICAL COEFFICIENT MATRIX

To provide each unit of the jth commodity the input used fort the ith commodity must be fixed amount which we shall denote by a_{ij}.

Thus $a_{ij} = a_{ij}/x_j$. If x_j denotes the total output of the jth commodity the input requirement of the ith commodity will be equal to $x_{ij} = a_{ij} \, x_j$.

We know determine the input output transaction table in terms of technical coefficients.

Purchasing Sector	Total output of the sector	Input requirements of producing sector			Requirements of final consumption
		x_1	x_2	x_3	
	Sales				
1	x_1	$a_{11}x_1$	$a_{12}x_2$	$a_{13}x_3$	d_1
2	x_2	$a_{21}x_1$	$a_{22}x_2$	$a_{23}x_3$	d_2
3	x_3	$a_{31}x_1$	$a_{32}x_2$	$a_{33}x_3$	d_3
Primary Input	L	L_1x_1	l_2x_2	l_3x_3	

It may be observed that all the input coefficients are non-negative (0). The above table determines the total output of each doctor in terms of technical; coefficients :

Thus we have

$$x_1 = a_{11}x_1 + \text{a12x2} + a_{13}x_3 + d_1$$

$$x_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + d_2$$

$$x_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + d_3$$

$$L = l_1x_1 + l_2x_2 + l_3x_3$$

The matrix form of above = n is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$X = Ax + ⨯,\ l = \sum_{i=1}^{3} L_i x_i$$

Or $\quad X(1 - A)$ ⨯

Where X and D are respectively the variable vectors and the final demand (constant term) vector. The matrix (1 – A) is called the technology matrix. If (1 – A) is non-singular them $(1–A)^{-1}$ can be found and we have th solution.

$$X = (1 - A)^{-1} ⨯.$$

16.6 HAWKIN'S SIMON CONDITION

In some situations input matrix solution provides outputs expressed by negative number. When our solution provides negative output, is signifies that more than one tonne (or any unit) of that product is used in the production of every one tonne of that product, which is not a creat situation.

In such situations Hawkin's Simon conditions help us. Our basic equation is X = (1A). ⨯ . The order that this may not give negative number as solution, the matrix (1 –A) should be such that

(i) The determinent of the matrix (1 – A) should always be positive.

(ii) The diagonal elements $1 - a_{11}$, a_{22} ... a_{nn} should be positive, that is a_{11}, a_{22} ... a_{nn} should be less than one.

One that of output of any sale should not employ more than / unit each of its own output.

These are known as Hawkin's Simson condition.

For the two industry case, the leontief matrix is

$$1 - A = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ a_{21} & 1- a_{22} \end{bmatrix}$$

The first condition will be

$$1 - a_{11} > 0$$

Second condition will be

$$|1 - A| > 0$$

If these condition satisfies, them it satisfies. Hawkin Simon conditions.

Q. Solved Exercise

In an economy there are two industries agriculture and manufacturer and the following tables gives the supply and their demand position.

	Agriculture	Manufacturer	Final demand	Total demand
Agriculture	75	125	100	300
Manufacturer	100	150	250	500

If final demand increases to 150 for Agriculture and 300 for manufacturer.

Find the total output required to meet these demand.

Sol. Here,

$$x_{11} = 75, \qquad x_{12} = 125, \qquad x_1 = 300$$

$$x_1 = 100, \qquad x_{22} = 150, \qquad x_2 = 500$$

$$a_{11} = \frac{x_{11}}{x_1} = \frac{75}{300} = \frac{1}{4}$$

$$a_{12} = \frac{x_{12}}{x_3} = \frac{125}{500} = \frac{1}{4}$$

$$a_{21} = \frac{x_{21}}{x_1} = \frac{100}{300} = \frac{1}{3}$$

$$a_{22} = \frac{x_2}{x_2} = \frac{150}{500} = \frac{3}{10}$$

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{10} \end{bmatrix} \text{ and } D = \begin{bmatrix} 150 \\ 300 \end{bmatrix}$$

$$[I - A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{10} \end{bmatrix}$$

$$[I - A] = \begin{bmatrix} 1 - \frac{1}{4} & 0 - \frac{1}{4} \\ 0 - \frac{1}{3} & 1 - \frac{3}{10} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{3} & \frac{7}{10} \end{bmatrix}$$

$$|I - A| = \frac{21}{40} - \frac{1}{12}$$

$$= \frac{63-10}{120} = \frac{53}{120} > 0$$

$$\therefore X = [I - A]^{-1} . D$$

$$[I - A]^{-1} = \frac{53}{120}\begin{bmatrix} \frac{7}{10} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix}$$

$$\therefore \text{Final demand} = \frac{1}{\frac{53}{120}}\begin{bmatrix} \frac{7}{10} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 150 \\ 300 \end{bmatrix}$$

$$= \frac{1}{\frac{53}{120}}\begin{bmatrix} \frac{7}{10}\times 150 + \frac{1}{4}\times 300 \\ \frac{1}{3}\times 150 + \frac{3}{4}\times 300 \end{bmatrix}$$

$$\Rightarrow \frac{1}{\frac{53}{120}}\begin{bmatrix} 7\times 15 + 75 \\ 50 + 3\times 75 \end{bmatrix}$$

$$\Rightarrow \frac{1}{\frac{53}{120}}\begin{bmatrix} 105 + 75 \\ 50 + 225 \end{bmatrix}$$

$$\Rightarrow \frac{1}{\frac{53}{120}}\begin{bmatrix} 180 \\ 275 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{120}{53}\times 180 \\ \frac{120}{53}\times 275 \end{bmatrix}$$

$$\begin{bmatrix} \text{Agriculture} \\ \text{Manufacturer} \end{bmatrix} \Rightarrow \begin{bmatrix} 407.54 \\ 622.64 \end{bmatrix}$$

  Now, final demand will be

Agriculture = 407.54

Manufacture = 622.64

Q. $[I - A] = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.2 & 1 & -0.5 \\ -0.4 & 0 & 1 \end{bmatrix}$**, labour coefficients are 0.4, 0.7, 1.2**

final demand D = $\begin{bmatrix} 1000 \\ 5000 \\ 4000 \end{bmatrix}$

Determine the level of output and employment. If wage rate is Rs. 10, per labour day ? Find the equilibrium prices ?

Ans. $[I - A] = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.2 & 1 & -0.5 \\ -0.4 & 0 & 1 \end{bmatrix}$

$|1 - A| = 0.8$

$$[1 - A]^{-1} = \frac{1}{0.8}\begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.2 & 0.9 \end{bmatrix}$$

$$X = [I - A]^{-1} D$$

$$= \frac{1}{0.8}\begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 1000 \\ 5000 \\ 4000 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5625 \\ 9250 \\ 6250 \end{bmatrix}$$

$$x_1 = 5625,\ x_2 = 9250,\ x_3 = 6250$$

Total labour requirement

$= \square_1 x_1 + \square_2 x_2 + \square_3 x_3$

$= (0.4)\ (5625) + (0.7)\ (9250) + (1.2)\ (6250)$

$= 16225$ (labour employment)

When wage rate is Rs. 10 per labour day, then

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \frac{1}{0.8} \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 0.4 \times 10 \\ 0.7 \times 10 \\ 1.2 \times 10 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \text{Rs. } 13.125 \\ \text{Rs. } 18.25 \\ \text{Rs. } 17.25 \end{bmatrix}$$

$p_1 =$ Rs. 13.125,

$p_2 =$ Rs. 18.25

$p_3 =$ Rs. 17.25

Q. Let A = $\begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$**, final demand** $d = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$

□ primary input =

$a_{01} = 0.3$, $a_{02} = 0.3$ and $a_{03} = 0.4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (I - A)d$$

$x_1 = 24.84$
$x_2 = 20.68$
$x_3 = 18.36$

□ required primary input = 21.00 billion

16.7 SUMMARY

We end this lesson by summarizing what we have covered in it :

i) Input-Output Analysis

ii) Simon-Hawkin conditions

16.7 LESSON AND EXERCISE

Q. 1. Define Input-Output Analysis ?

Q. 2. Describe Input-Output transaction table ?

Q. 3. What is Hawkins-Simon Condition

Q. 4. Suppose there are two sectors A, B and the final demand F. The input output table is given as follows :

Sectors	A	B	F	Total Output
A	15	20	45	80
B	5	20	15	40

What will be the level of output if the final demand becomes 65 for A and 25 for B.

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